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A symplectic approach to certain functional integrals and partition functions

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Abstract

We define the Liouville functional on the set of functions on an infinite-dimensional symplectic manifold which are Hamiltonian with respect to a torus-action. In the case of finite-dimensional manifolds this functional is closely connected with the integral over the Liouville measure by a theorem due to Duistermaat and Heckman. The symplectic setup turns out to be natural for the calculation of partition functions of certain quantum field theories. In particular, among other examples, we calculate the partition function of the Wess–Zumino–Witten model on an elliptic curve in terms of this functional and deduce its modular invariance from its expression as a functional integral. In the case that the symplectic manifold is given as a generic coadjoint orbit of a loop group, the Liouville functional can be shown to give the same result as usual integration with respect to the Wiener measure. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Functional integration methods play an important role in modern quantum field theory (cf. e.g. [9,26]). Despite this fact, the precise mathematical meaning of the appearing integrals remains, at least at this time, rather mysterious. Only in certain situations, one can rely on well developed theories such as the Wiener measure, but usually functional integrals are "calculated" by ad hoc methods which are justified merely by their analogy with finite-dimensional integration methods. Surprisingly enough, quite often these calculations yield results which can also be derived without using functional integration. Two

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examples of such "calculations" are the papers [1,19] in which the Duistermaat Heckman exact integration formula is applied to integrals on certain infinite-dimensional symplectic manifolds.

Let us briefly review these results: The Duistermaat Heckman formula applies to the computation of integrals of the form $\int_M \mathrm{e}^{-tf}(\omega^n/n!)$, where (M, ω) is a finite-dimensional compact symplectic manifold of dimension $2n$, and $\omega^n/n!$ the associated Liouville measure. Let us assume that the circle S^1 acts on M preserving ω and that f is a Hamiltonian function corresponding to the vector field generated by the S^1-action. If the fixed point set of the S^1-action is discrete, the Duistermaat Heckman formula [3] allows to reduce the integral $\int_M \mathrm{e}^{-tf}(\omega^n/n!)$ to a sum over the fixed points of the circle action (see Section 2.1 for an exact statement of the formula). Following ideas of Witten, Atiyah [1] indicated how applying the Duistermaat Heckman formula to the loop space of a Riemannian manifold which has a (degenerate) two-form and a natural S^1-action, one can formally derive the index theorem for the Dirac operator. Perret [19] used the Duistermaat Heckman theorem on the loop space of a coadjoint orbit of a compact Lie group, resp. a loop group to give "physical proofs" of the Weyl and the Kac–Weyl character formulas, respectively.

In the present paper, we will use similar ideas to calculate certain integrals over infinite-dimensional symplectic manifolds naturally arising in the theory of loop groups and double loop groups. But our approach to these "integrals" will be somewhat more conceptual than the one employed in the physics literature. In particular, instead of "calculating" the integrals using the Duistermaat Heckman formula, we will use an analogue of the Duistermaat Heckman formula to define a functional on the Hamiltonian functions corresponding to some symplectic torus-action on M. We will call this functional the Liouville functional because of its analogy with the Liouville measure on a finite-dimensional manifold. By comparing the symplectic form ω with a Riemannian structure σ on M, we define a second functional. In the finite-dimensional case, this second functional is equal to the integration over the Riemannian volume form. Thus, by analogy, we will refer to this functional as "integration" over the Riemannian volume form $\mathrm{d}\sigma$. This will be done in Section 2.

In Section 3, we will "integrate" several functions on the coadjoint orbit of the centrally extended loop group $\hat{G}$ of a compact Lie group G with respect to the Riemannian volume form. The resulting functions on G arise naturally in the representation theory of G and its associated affine Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ and can be interpreted as the partition function of a quantum mechanical particle moving on the group G. In [7], Frenkel gave an interpretation of the same functions in terms of integrals with respect to the Wiener measure on a completion of the corresponding coadjoint orbit of $\hat{G}$. We will compare these two approaches in Section 3.4 and see that they are, in a sense, equivalent. Therefore, at least in these cases the name "integral" for our functionals is justified not only by analogy.

Section 4 is devoted to the calculation of the partition function of the gauged Wess–Zumino–Witten (WZW) model on an elliptic curve using the Liouville functional approach. The WZW model is a quantum field theory on a compact Riemann surface Σ with values in a compact Lie group G, or more generally in its complexification $G_{\mathbb{C}}$. We will only consider the case in which Σ is the elliptic curve $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, where $\tau \in \mathbb{C}$, $\mathrm{im}(\tau) > 0$ is the modular parameter of the elliptic curve. The partition function of the gauged WZW model at level $\kappa \in \mathbb{N}$ is formally given by the functional integral

$$\int_{C^\infty(\Sigma_\tau, G_\mathbb{C})} \mathrm{e}^{-S_{G,H,\kappa}(g)} \mathcal{D}(g),$$

where H is an element of the Lie algebra of G, and $S_{G,H,\kappa} : C^\infty(\Sigma_\tau, G_\mathbb{C}) \to \mathbb{R}$ is the so-called gauged WZW action (see Section 4.1 for more details and references). At this time, the measure theoretic meaning of this integral is not clear (but cf. Section 4.4 for some speculations). In any case, let T be a maximal torus of G such that $\exp(H) \in T$ and let $T_\mathbb{C} \subset G_\mathbb{C}$ be its complexification. Assume that $\exp(H)$ is a regular element of G (i.e. the maximal torus T such that $\exp(H) \in T$ is unique). The action function $S_{G,H,\kappa}$ factors through $LLG_\mathbb{C}/T_\mathbb{C}$, where $LLG_\mathbb{C}$ denotes the double loop group of $G_\mathbb{C}$ (i.e. the space of all smooth maps from $S^1 \times S^1$ to $G_\mathbb{C}$). Now $LLG_\mathbb{C}/T_\mathbb{C}$ has a (complex valued) symplectic form and a natural action of the torus $S^1 \times S^1 \times T$, where the first two factors act by rotating the loops and T acts by left multiplication. We will show that the gauged WZW action is the Hamiltonian of a vector field defined by this torus-action. Since the torus-action has a discrete fixed point set, we can calculate the "integral"

$$\int_{LLG_\mathbb{C}/T_\mathbb{C}} \mathrm{e}^{-S_{G,H,\kappa}} = c \sum_{\lambda \in \tilde{P}_+^k} |\chi_\lambda(\tau, H)|^2$$

with some $c \in \mathbb{R}$. Here P_+^k denotes the set of highest weights at some level k of the untwisted affine Lie algebra $\tilde{\mathfrak{g}}_\mathbb{C}$ corresponding to G and χ_λ denotes the corresponding affine character (see Theorem 4.5).

It is well known that the sum above is invariant under a certain $SL(2, \mathbb{Z})$-action (cf. [13]). Since the holomorphic structure on the torus $S^1 \times S^1$ defining the elliptic curve $\sum_\tau$ stays invariant under the natural $SL(2, \mathbb{Z})$-action on $S^1 \times S^1$, we can use the functional integral approach developed in this paper to deduce the modular invariance of the function

$$\sum_{\lambda \in \tilde{P}_+^k} |\chi_\lambda(\tau, 0)|^2.$$

We close this exposition with a remark on a "twisted" version of the WZW model and some speculations about a measure theoretic interpretation of the calculations leading to the partition function of the WZW model.

2. The Liouville functional

2.1. The Liouville functional

Let (M, ω) be a finite-dimensional compact symplectic manifold of dimension $2n$. The Liouville volume form associated with the symplectic form ω is the $2n$-form $\omega^n/n!$. Let us assume that we have an action of some torus $T = \mathbb{R}^l/\mathbb{Z}^l$ on M which preserves ω. The Lie algebra of T will be denoted by $\mathfrak{h}$. We will call $H \in \mathfrak{h}$ a generic element of $\mathfrak{h}$ if the group generated by $\exp(H)$ is a dense subgroup of T. Any H defines an $\mathbb{R}$-action and thereby a vector field $\tilde{H}$ on M. Assume that J_H is a Hamiltonian function corresponding to this $\mathbb{R}$-action. That is, we have the identity $\mathrm{d}J_H = \iota_{\tilde{H}}\omega$. Furthermore, assume that T acts effectively on M and that the fixed point set P consists of isolated points $p \in P$. Then T acts linearly on the tangent spaces T_pM. After picking an almost complex structure on M which is compatible with the symplectic form ω and which commutes with the T-action, we have a decomposition $T_pM = \oplus_{j=1}^n V_j^p$ into (complex) one-dimensional representations V_j^p of T. Here T acts on V_j via the complex character $t \mapsto \exp(2\pi \mathrm{i}\alpha_j^p(H))$, where $\exp(H) = t$.

Now we can state the Duistermaat Heckman exact integration formula [3].

Theorem 2.1. *Let M, ω, H, P be as above. Then*

$$\int_M \mathrm{e}^{-tJ_H}\frac{\omega^n}{n!} = \sum_{p\in P}\frac{\mathrm{e}^{-tJ_H(p)}}{t^n\prod_{j=1}^n \alpha_j^p(H)},$$

where t can be a real or complex parameter.

This theorem can be easily extended to the case when the fixed point set of the T-action consists of sub-manifolds instead of isolated points (cf. [1,3,4]).

In the case that M has a Riemannian metric σ the associated Riemannian volume form $\mathrm{d}\sigma$ and the Liouville form on M are related via

$$\frac{\omega^n}{n!} = \mathrm{Pf}(B_\sigma)\,\mathrm{d}\sigma,$$

where B_σ is the skew symmetric endomorphism of the tangent bundle associated to ω by the metric σ (i.e. $\omega_x(X, Y) = \sigma_x(B_{\sigma,x}(X), Y)$ for $X, Y \in T_xM$), and Pf is the Pfaffian.

If the manifold M is infinite-dimensional, the Liouville measure does not make sense. Ignoring this fact, physicists use the Duistermaat Heckman formula to "calculate" certain integrals over the Liouville measure (see e.g. [1,19]). We will proceed in the opposite direction and use an analogue of the Duistermaat Heckman formula to define a functional on the set of functions on M which are Hamiltonian with respect to the T-action on M. We will call this functional the Liouville functional because of its analogy with the Liouville measure on a finite-dimensional manifold.

To get started, let (M, ω) be an infinite-dimensional symplectic manifold. That is, M is a Frechet-manifold together with a closed two-form ω which is non-degenerate in the sense that the map $T_mM \to T_m^*M$, $X_m \mapsto \omega_m(X_m, \cdot)$ is injective at each $m \in M$. Furthermore, we assume the tangent spaces T_mM to have a countable basis for all $m \in M$. Finally, we have to make the further assumption that (M, ω) admits a compatible almost complex structure, that is an endomorphism I of TM such that $I^2 = -1$, $I^*\omega = \omega$ and $\omega(\cdot, I\cdot)$ is positive definite.

Now suppose there is an effective action of a torus T on M which preserves ω and I. Let us assume that the fixed point set P of the T-action consists of (possibly infinitely many) isolated points $p \in P$. Then we have a T-action on the tangent spaces T_pM which again decompose into the direct sum of complex one-dimensional representations of T. That is, $T_pM = \oplus_{j\in\mathbb{N}} V_j^p$ where, as before, T acts on V_j^p via a complex character $\exp(H) \mapsto \exp(2\pi \mathrm{i}\alpha_j^p(H))$ for $H \in \mathfrak{h}$. Now we have $\alpha_j^p(H) \in \mathbb{R}$ for all $p \in P$ and $j \in \mathbb{N}$. Let us assume that the series $\{|\alpha_j^p(H)|\}_{j\in\mathbb{N}}$ is zeta-multipliable for each $p \in P$ and denote the corresponding zeta-function by ζ. (See Appendix A for a brief introduction to the theory of zeta-regularized products.) Then we set

$$Z_p(H) = \left(\prod_{j\in\mathbb{N}} |\alpha_j^p(H)| \right)_\zeta .$$

Up to sign, this is the infinite-dimensional analogue of the denominator in the Duistermaat Heckman formula for finite-dimensional compact manifolds. To take care of the sign, we define $\#p$ to be the number of rotation planes V_j^p for which $\alpha_j^p(H) < 0$. We have to make the assumption that $\#p$ is finite for all $p \in P$. Then $(-1)^{\#p}$ will be the desired sign.

We have collected all the necessary structures to define the Liouville functional.

Definition 2.2. Let (M, ω) be an infinite-dimensional symplectic manifold with a T-action for which all the assumptions above are satisfied. For $H \in \mathfrak{h}$ as above, let J_H be a Hamiltonian function of the $\mathbb{R}$-action on M defined by H. Then for $t \in \mathbb{R}_{>0}$ we define the Liouville functional $L_t(J_H)$ via

$$L_t(J_H) = \sum_{p\in P} (-1)^{\#p} \frac{\mathrm{e}^{-tJ_H(p)}}{t^{\zeta(0)} Z_p(H)},$$

whenever this sum makes sense.

Note that L_t is a non-linear functional.

Let us assume, that additionally to being symplectic, M is a Riemannian manifold with Riemannian metric σ. That is, we have a non-degenerate symmetric bilinear form $\sigma_m = \langle\cdot,\cdot\rangle_m$ on each tangent space $T_m M$ which varies smoothly with m. As before, by non-degeneracy we mean that the induced map $T_m M \to T_m^* M$ is injective. In this case, we can define an analogue to the integration with respect to the Riemannian volume form on M provided that ω and σ are compatible in the following sense.

Let us assume, there exists a skew symmetric automorphism $B_{\sigma,x}$ of $T_x M x \in M$ for each $x \in M$ such that

$$\omega_x(X, Y) = \sigma_x(B_{\sigma,x}(X), Y).$$

Furthermore, let us assume that the zeta-regularized determinant $\det_\zeta(B_{\sigma,x})$ (i.e. the zeta-regularized product of the eigenvalues of $B_{\sigma x}$) exists. Now, if the zeta-regularized determinant defines a nowhere vanishing positive function on M, we can define a function $\mathrm{Pf}_\zeta(B_\sigma) : M \to \mathbb{R}_+$ such that $\mathrm{Pf}(B_\sigma)(x)^2 = \det_\zeta(B_{\sigma,x})$. This function will be called the zeta-regularized Pfaffian of B_σ. We will call the symplectic form ω and the Riemannian metric σ zeta-compatible if such $\mathrm{Pf}_\zeta(B_\sigma)$ exists. On a finite-dimensional manifold, ω and σ are compatible, exactly if they define the same orientation of M, and $\mathrm{Pf}(B_\sigma)$ is the function relating the two $2n$-forms.

In analogy with the finite-dimensional case, we can now set

$$\int_M \mathrm{e}^{-tJ_H} \mathrm{Pf}_\zeta(B_\sigma)\, \mathrm{d}\sigma = L_t(J_H).$$

In the cases we will be considering, M will be a homogeneous space $M = G/G'$ and ω and σ can be chosen invariant under the canonical G-action on M. In this case, if $\mathrm{Pf}_\zeta(B_\sigma)$ exists, it will be a constant. Therefore, such an M is always orientable and we have

$$\int_M \mathrm{e}^{-tJ_H}\,\mathrm{d}\sigma = \frac{L_t(J_H)}{\mathrm{Pf}_\zeta(B_\sigma)(x_0)}.$$

2.2. The Liouville functional on a complex manifold

In this section, we will describe how one can extend the formalism developed in the last section to the case when the manifold M is complex and $\omega = \omega_1 + \mathrm{i}\omega_2$ is a closed non-degenerate complex valued $\mathbb{C}$-linear two-form. In this case ω will not be compatible with the natural complex structure I on TM which is given by multiplication with i, since we have $\omega(IX, IY) = -\omega(X, Y)$. So we have to assume that TM admits a second complex structure J which anti-commutes with I and is compatible with ω in the following sense: $J^*\omega = \omega$, and $\omega(\cdot, J\cdot)$ is “positive definite” in the sense that for all $m \in M$, $X_m \in T_m M$ we have either $\omega_2(X_m, J_m(X_m)) > 0$, or $\omega_2(X_m, J_m(X_m)) = 0$ and $\omega_1(X_m, J_m(X_m)) > 0$.

As in the preceding section, let us assume that we have an action of some torus T on M which leaves the symplectic form ω and the two complex structures I and J invariant, and which has a discrete fixed point set P. Thus, J gives the tangent spaces T_pM the structure of quaternionic representations of T (see [2]). Now we can decompose each $T_pM = T_pM^+ \oplus T_pM^-$, where $T_pM^\pm$ denotes the $\pm$i eigenspace of J. The spaces T_pM^+ and T_pM^- are isomorphic as vector spaces.

We can decompose $T_pM^+ = \oplus_{j\in\mathbb{N}} V_j^p$, into a direct sum of one-dimensional complex representations (with respect to the complex structure J), such that T acts on V_j^p via the character $\exp(2\pi\mathrm{i}\alpha_j^p)$. With these choices made, we can define $Z_p(H)$, $\#p$, and the Liouville functional $L_t(J_H)$ of some Hamiltonian function J_H exactly as in Section 2.1.

Finally, in the complex setting it is natural and in fact necessary for applications, to allow the element H considered above to be in the complexified Lie algebra $\mathfrak{h}_\mathbb{C}$ of T rather than in the real Lie algebra $\mathfrak{h}$. Since M is complex, such H defines a vector field $\tilde{H}$ on M, and we shall call a function $J_H : M \to \mathbb{C}$ Hamiltonian with respect to $\tilde{H}$ if $\mathrm{d}J_H = \iota_{\tilde{H}}\omega$. The formalism of the Liouville functional can be generalized to this setting without major changes. We get a decomposition of the tangent spaces $T_pM = \oplus_{j\in\mathbb{N}} V_j^p$ into complex (with respect to I) one-dimensional spaces V_j^p on which the Abelian Lie algebra $\mathfrak{h}_\mathbb{C}$ acts via the character $2\pi\mathrm{i}\alpha_j^p$. The only difference to the real case is that now the $\alpha_j^p(H)$ might not be in $\mathbb{R}$, which causes problems with the zeta-regularized products and the definition of the number $\#p$ appearing in the definition of the Liouville functional. In the examples we will consider, we will be able calculate the zeta-regularized products using a trick. So the only thing, we have to take care about is the definition of the number $\#p$ in this more general setting.

To generalize the definition of $\#p$, let us first take a closer look at what happened in the case that $H \in \mathfrak{h}$: We have decompositions of the tangent spaces of M at p into four-dimensional real representations $T_pM = \oplus_j (V_{\alpha_j}^p \oplus V_{-\alpha_j}^p)$ of T, such that if one diagonalizes the T-action with respect to the complex structure I, the torus acts on $V_{\alpha_j}^p$ via the character $\exp(2\pi\mathrm{i}\alpha_j^p)$.

In this setting, the complex structure J defines an $\mathbb{R}$-linear map $J : V_\alpha^p \to V_{-\alpha}^p$. Restricting the T-action to the $+\mathrm{i}$ eigenspace of J amounts to picking one character out of each pair $\pm\alpha_j^p$ appearing in the decomposition of T_pM. Now the choice of some regular $H \in \mathfrak{h}$ (i.e. $\beta(H) \notin \mathbb{Z}$ for all characters $\exp(2\pi\mathrm{i}\beta)$ of T), gives a decomposition of the character lattice Q of T into positive and negative characters via declaring $\beta \in Q$ positive if $\beta(H) > 0$. In this picture, $\#p$ is exactly the number of negative characters appearing in the series $\{\alpha_j^p\}_{j\in\mathbb{N}}$.

Now it is straightforward to generalize our definition of $\#p$ to the complex case: The element $H \in \mathfrak{h}_\mathbb{C}$ comes from an infinitesimal action of the complexified torus $T_\mathbb{C}$ on M. So we have to pick a decomposition of the character lattice $Q\backslash\{0\} = Q_+ \cup Q_-$ of $T_\mathbb{C}$ into positive and negative characters. In analogy with the real case, this decomposition should be defined by the element $H \in \mathfrak{h}_\mathbb{C}$: We define $\alpha \in Q$ to be positive, if either $\mathrm{Im}(\alpha(H)) > 0$, or $\mathrm{Im}(\alpha(H)) = 0$ and $\mathrm{Re}(\alpha(H))$. This choice of decomposition of the character lattice Q into a positive and negative part agrees with the definition of "positive definiteness" of the $\mathbb{C}$-valued symmetric bilinear form $\omega(\cdot, J\cdot)$ above. Then, as before, we can set $\#p$ to be the number of negative characters appearing in the series $\{\alpha_j^p\}_{j\in\mathbb{N}}$.

3. The partition function for compact Lie groups

3.1. An infinite-dimensional flag manifold

Let us look at a first example in which the Liouville functional can be calculated and gives rise to an interesting function: Consider the infinite-dimensional manifold LG/T, where G is a compact semi-simple simply connected Lie group with maximal torus T and LG denotes the loop group of G in the sense of [21]. That is, $LG = C^\infty(S^1, G)$ with pointwise multiplication. The Lie algebra of LG is $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. Let $\langle\cdot,\cdot\rangle$ denote the negative of the Killing form on $\mathfrak{g}\otimes\mathbb{C}$. This gives a positive definite G-invariant bilinear form on $\mathfrak{g}$. We can use $\langle\cdot,\cdot\rangle$ to define an antisymmetric bilinear form ω on $L\mathfrak{g}$ via

$$\omega(X, Y) = \int_{t=0}^{1} \langle X'(t), Y(t)\rangle\,\mathrm{d}t,$$

where we have parametrized the circle via $t \in [0, 1]$. Obviously, this form is degenerate exactly in the space of constant loops. Thus, using the G-invariance of $\langle\cdot,\cdot\rangle$, it gives rise to a symplectic form ω on LG/G via left translation. Also, G/T is a generic coadjoint orbit of G and hence has a symplectic structure for each generic $H \in \mathfrak{h}$ given by the Kirillov form ω_0^H. On the tangent space at eT, this form is given by

$$\omega_0^H(X, Y) = \langle H, [Y, X]\rangle.$$

As a manifold, LG/T is isomorphic to $LG/G \times G/T$. So for each generic $H \in \mathfrak{h}$, we have a symplectic form on LG/T given by $\omega^H = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega_0^H$.

There is a second symplectic structure on LG/T which comes from the fact that LG/T is isomorphic to a coadjoint orbit of the group LG. The coadjoint action of $\gamma \in LG$ on $L\mathfrak{g}\oplus\mathbb{R}$ is given by (cf. [7,21])

$$\gamma(X, \lambda) = (\mathrm{Ad}\,\gamma(X) + \lambda\gamma'\gamma^{-1}, \lambda).$$

Here $L\mathfrak{g}$ is identified with the smooth part of $(L\mathfrak{g})^*$ via the non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $L\mathfrak{g}$ defined by

$$\langle X, Y\rangle = \int_0^1 \langle X(t), Y(t)\rangle \,\mathrm{d}t.$$

The orbits of this action can be classified in terms of conjugacy classes of the corresponding compact Lie group G via the following construction: For each $(X, \lambda) \in L\mathfrak{g} \oplus \mathbb{R}$ with $\lambda \neq 0$ one can solve the differential equation

$$z' = -\lambda^{-1} X z$$

with initial condition $z(0) = 1$. Since X is periodic in t, we have $z(t+1) = z(t)M_X$, where $M_X = z(1) \in G$ is the monodromy of the differential equation. Now the theory of differential equations with periodic coefficients (cf. [7,21]) implies the following proposition.

Proposition 3.1.

1. *For $\lambda \neq 0$, the orbits of LG on $L\mathfrak{g} \times \{\lambda\}$ correspond precisely to the conjugacy classes of G under the map $(X, \lambda) \mapsto M_X$.*
2. *The stabilizer of (X, λ) in LG is isomorphic to the centralizer Z of M_X in G under the map $\gamma \mapsto \gamma(0)$.*

Let H be a generic element of $\mathfrak{h}$ and $\lambda \neq 0$. From Proposition 3.1 we see that the orbit of LG through (H, λ) is isomorphic to LG/T. The Kirillov form $\tilde{\omega}^H$ on LG/T is defined exactly as in the finite-dimensional case. Now we can compare the two symplectic forms on LG/T.

Lemma 3.2. *The two-forms ω^H and $-\tilde{\omega}^H$ lie in the same cohomology class of LG/T.*

Proof. This is a direct generalization of [21, Proposition 4.4.4]. □

It was shown in [21, Chapter 8.9] that the symplectic manifold LG/G admits a complex structure which makes it into a Kähler manifold. Furthermore, we can pick a complex structure on G/T which is compatible with the symplectic form ω_0^H. Putting these two structures together, we get a complex structure on LG/T which is compatible with ω^H.

Our next goal is to show that the symplectic form ω^H and the Riemannian metric σ on M are zeta-compatible in the sense of Section 2.1. The Riemannian metric on LG/T is given by

$$\sigma_{eT}(X, Y) = \int_0^1 \langle X(t), Y(t)\rangle \,\mathrm{d}t,$$

while the symplectic form ω^H is given by

$$\begin{aligned}\omega^H_{eT}(X, Y) &= \int_0^1 \langle X'(t), Y(t)\rangle + \int_0^1 \langle H, [Y(t), X(t)]\rangle \,\mathrm{d}t\\ &= \sigma_{eT}(X', Y) - \int_0^1 \langle [H, X(t)], Y(t)\rangle \,\mathrm{d}t.\end{aligned}$$

So the endomorphism $B_{\sigma,eT}$ is given by

$$B_{\sigma,eT}=\frac{\partial}{\partial t}-\mathrm{ad}(H).$$

Since both the Riemannian metric and the symplectic form are defined on LG/T via left translation, the zeta-regularized Pfaffian of B_σ — if it exists — will be a constant. So to check the compatibility of ω and σ, we have to show that the zeta-regularized Pfaffian $\mathrm{Pf}_\zeta(B_\sigma)(eT)$ indeed exists. Let us identify the root system Δ of $\mathfrak{g}\otimes\mathbb{C}$ with a subset of the character lattice Q of T defined above (see e.g. [2]).

Lemma 3.3. *The Pfaffian* $\mathrm{Pf}_\zeta(B_\sigma)(eT)$ *exists and is given by*

$$\mathrm{Pf}_\zeta(B_\sigma)(eT)=(2\pi)^{\dim\mathfrak{g}}\prod_{\alpha\in\Delta_+}2\sin(\pi\alpha(H)),$$

where Δ_+ *is the set of positive roots with respect to the Weyl chamber* $K\subset\mathfrak{h}$ *such that* $H\in K$.

Remark 3.4. Note that $\prod_{\alpha\in\Delta_+}2\sin(\pi\alpha(H))$ is the denominator in the Weyl character formula for the compact Lie group G.

Proof. Using the root space decomposition of $\mathfrak{g}\otimes\mathbb{C}$, one sees that the eigenvalues of $B_{\sigma,eT}$ are $\{\pm 2\pi\mathrm{i}n\}_{n\in\mathbb{N}}\cup\{\pm 2\pi\mathrm{i}n\pm 2\pi\mathrm{i}\alpha(H)\}_{n\in\mathbb{N}_0,\alpha\in\Delta_+}$. The multiplicity of the eigenvalues is 1 if $\alpha\neq 0$ and $l=\dim T$ if $\alpha=0$. Thus, the zeta-regularized determinant of $B_{\sigma,eT}$ is

$$\begin{aligned}\det_\zeta(B_{\sigma,eT})&=\prod_{\alpha\in\Delta_+}\left((2\pi\alpha(H))^2\prod_{n=1}^{\infty}(2\pi)^4(n^2-\alpha(H)^2)^2\right)_\zeta\left(\prod_{n=1}^{\infty}(2\pi n)^2\right)_\zeta^l\\&=\prod_{\alpha\in\Delta_+}\left(2\pi\alpha(H)\prod_{n=1}^{\infty}(2\pi n)^2\left(1-\frac{\alpha(H)^2}{n^2}\right)\right)_\zeta^2\left(\prod_{n=1}^{\infty}2\pi n\right)_\zeta^{2l}\\&=\prod_{\alpha\in\Delta_+}4\sin^2(\pi\alpha(H))\left(\prod_{n=1}^{\infty}2\pi n\right)_\zeta^{2\dim\mathfrak{g}}=(2\pi)^{2\dim\mathfrak{g}}\prod_{\alpha\in\Delta_+}4\sin^2(\pi\alpha(H)),\end{aligned}$$

where we have used the identity

$$\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2n^2}\right)=\frac{\sin(x)}{x},$$

and the first example for zeta-regularized products from Appendix A. Since the Pfaffian $\mathrm{Pf}_\zeta(B_\sigma)(eT)$ is defined as the square root of $\det_\zeta(B_{\sigma,eT})$, the lemma follows. □

3.2. A torus-action on LG/T

Let us identify LG/G with the space of based loops $\Omega G = \{\gamma \in LG | \gamma(0) = e\}$. The circle S^1 acts on ΩG by rotations R_t, where

$$R_t\gamma(u) = \gamma(u+t)\gamma(t)^{-1}.$$

The maximal torus $T \subset G$ acts by conjugation on LG/G, so together we get an $S^1 \times T$-action: An element $(t, \exp(H)) \in S^1 \times T$ acts on LG/G via

$$(t, \exp(H)) : \gamma \mapsto \exp(tH) R_t(\gamma) \exp(-tH).$$

The fixed points of this action are precisely the homomorphisms $\gamma : S^1 \to T$ ([21, Chapter 8.9]). Furthermore, T acts by left multiplication on G/T with fixed point set $N(T)/T$, where $N(T)$ is the normalizer of T in G. Letting S^1 act trivially on G/T, we get an $S^1 \times T$-action on LG/T which has fixed point set $Q^\vee \times W$, where $Q^\vee$ denotes the lattice of homomorphisms $S^1 \to T$ and $W = N(T)/T$ is the Weyl group of G. It is straightforward to check that this torus-action leaves the symplectic form ω^H as well as the complex structure I invariant.

Take a generic $H \in \mathfrak{h}$. Such H defines an $\mathbb{R}$-action on LG/T by the construction outlined above. If $tH \in \ker(\exp)$ for some $t \in \mathbb{R}^*$, we get in fact an S^1-action but this will be of no concern for us. In both cases, the fixed point set of this $\mathbb{R}$-action is still $Q^\vee \times W$.

Now let us compute the denominator $z_p(H)$ of the Liouville functional: The tangent space of LG/T at the point eT is isomorphic to $L\mathfrak{g}/\mathfrak{h}$. Its decomposition into rotation planes is exactly the decomposition of $L\mathfrak{g}/\mathfrak{h}$ into eigenspaces of the endomorphism $\beta_\sigma(eT)$ used in Lemma 3.3. As we saw in the proof of Lemma 3.3, the eigenvalues of the torus-action are given by the two series $\{2\pi \mathrm{i}(\pm\alpha(H) \pm n)\}_{\alpha\in\Delta_+, n\geq 0}$ and $\{\pm 2\pi \mathrm{i} n\}_{n>0}$ again with the multiplicity 1 if $\alpha = 0$ and l if $\alpha \neq 0$. Remember that in the definition of $Z_{eT}(H)$, the eigenvalues of the torus-action were multiplied by $1/2\pi$. Therefore, we have to calculate the regularized product

$$Z_{eT}(H) = \prod_{\alpha\in\Delta_+} \left((\alpha(H))^2 \prod_{n=1}^{\infty} (n^2 - \alpha(H)^2)^2 \right)_\zeta \left(\prod_{n=1}^{\infty} n^2 \right)_\zeta^l .$$

Now the same calculation as in the proof of Lemma 3.3 yields the following lemma.

Lemma 3.5.

$$Z_{eT}(H) = (\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H)).$$

So the series defining $Z_{eT}(H)$ is zeta-multipliable and we have checked all the necessary premises to calculate the Liouville functional of a Hamiltonian of our $\mathbb{R}$-action on M.

Finally, we have to calculate the number $\#(\beta, w)$ for the fixed points $(\beta, w) \in Q^\vee \times W$ of the torus-action. To do this, we will identify the fixed point set $Q^\vee \times W$ with the Weyl group $\tilde{W}$ of the untwisted affine Lie algebra $\tilde{\mathfrak{g}}_\mathbb{C}$ corresponding to the Lie algebra G. Furthermore, the set $\{\pm\alpha \pm n | \alpha \in \Delta_+, n \geq 0\} \cup \{\pm n | n > 0\}$ can be identified with the root system $\tilde{\Delta}$ of $\tilde{\mathfrak{g}}_\mathbb{C}$ (cf. [12]). Let us assume, that $H \in \mathfrak{h}$ lies in a fundamental alcove of the $\tilde{W}$-action on

$\mathfrak{h}$. Then the set $\tilde{\Delta}_+ = \{\alpha \in \tilde{\Delta} | \alpha(H) > 0\}$ defines a decomposition of $\tilde{\Delta}$ into positive and negative roots. Now one can see (for example by identifying the tangent spaces $T_{(\beta,w)}LG/T$ with $T_e LG/T$ via left multiplication by a representative of $(\beta, w)^{-1}$ in LG) that $\#(\beta, w)$ is exactly the number of positive roots of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ which are mapped to negative roots by the action of (β, w) on the root system of $\tilde{\mathfrak{g}}_{\mathbb{C}}$. By definition, this is the length $l(\beta, w)$ of (β, w) in $\tilde{W}$.

3.3. Calculation of a Liouville functional

Given $H \in \mathfrak{g}$ and $\gamma \in LG$, we can define a vector field $\mathrm{ad}\, H(\gamma)$ along γ via

$$\mathrm{ad}\, H(\gamma) = \left.\frac{\partial}{\partial s}\right|_{s=0} \exp(sH)\gamma \exp(-sH).$$

For generic $H \in \mathfrak{h}$ let us define a function $J_H : LG \to \mathbb{R}$ via

$$\gamma \mapsto \frac{1}{2}\int_0^1 \left\| \left(\frac{\partial}{\partial t}\gamma(t) - \mathrm{ad}\, H(\gamma(t)) \right) \gamma^{-1}(t) \right\|^2 \mathrm{d}t.$$

Since the scalar product $\langle \cdot, \cdot \rangle$ is G invariant, we have $J_H(\gamma) = J_H(\gamma h)$ for all $h \in T$. Therefore, J_H defines a function on LG/T which will be denoted with the same symbol. Suppose we have fixed a finite-dimensional faithful representation of the group G. Then we can write $J_H(\gamma) = \frac{1}{2}\int_0^1 \|\gamma'(t)\gamma^{-1}(t) + \gamma(t)H\gamma^{-1}(t) - H\|^2 \,\mathrm{d}t$.

Lemma 3.6. *The Hamiltonian vector field on LG/T corresponding to J_H is exactly the vector field on LG/T coming from the $\mathbb{R}$-action defined by H.*

Proof. This is a calculation similar to the proof of Proposition 8.9.3 in [21]. □

The next step is to calculate J_H in the fixed points of the $\mathbb{R}$-action: Let $g_w \in N(T)$ be a representative for $w \in W$, and let $\beta \in Q^\vee$. Then for $\gamma(t) = g_w \exp(t\beta)$ we have

$$J_H(\gamma) = \frac{1}{2}\int_0^1 \|\beta + w(H) - H\|^2 \,\mathrm{d}t = \frac{1}{2}\|\beta + w(H) - H\|^2.$$

Plugging this into the definition of the Liouville functional yields

$$L_1(J_H) = \frac{1}{(\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))} \sum_{w\in W}\sum_{\beta\in Q^\vee} (-1)^{l(w)}\, \mathrm{e}^{-(1/2)\|\beta+w(H)-H\|^2}.$$

Let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ denote the half sum of positive roots. A calculation due to Frenkel ([7, Theorem 4.3.4]) involving Poisson re-summation of the sum above, gives

$$\sum_{w\in W}\sum_{\beta\in Q^\vee} (-1)^{l(w)}\, \mathrm{e}^{-(1/2)\|\beta+w(H)-H\|^2} = \frac{\prod_{\alpha\in\Delta_+} 4\sin^2(\pi\alpha(H))}{(\sqrt{2\pi})^l \operatorname{vol}(Q^\vee)} \sum_{\lambda\in P_+} |\chi_\lambda(\exp(H))|^2\, \mathrm{e}^{-(1/2)\|\lambda+\rho\|^2},$$

where P_+ denotes the set of dominant weights of G and χ_λ the irreducible character of G corresponding to $\lambda \in P_+$.

Putting the above calculations together with the definition of the Riemannian volume $\mathrm{d}\sigma$ in Section 2.1 yields the main theorem for this section.

Theorem 3.7. *The following identity is valid*:

$$\int_{LG/T} \mathrm{e}^{-J_H}\,\mathrm{d}\sigma = \frac{1}{(2\pi)^{l+\dim \mathfrak{g}}\,\mathrm{vol}(Q^\vee)} \sum_{\lambda\in P_+} |\chi_\lambda(\exp(H))|^2\,\mathrm{e}^{-(1/2)\|\lambda+\rho\|^2}.$$

Remark 3.8. In our calculations leading to the partition function of the Lie group G, we always chose the element $H \in \mathfrak{h}$ which defines the symplectic structure on LG/T to be the same as the element $K \in \mathfrak{h}$ which defines the $\mathbb{R}$-action. Of course this is not necessary and was only done to emphasize the similarity of the calculations with those leading to the partition function of the WZW model in Section 4.3. Indeed, for certain choices of H and K, the resulting function will have a very natural interpretation as we shall see momentarily.

Choose $a \in \mathbb{R}_{>0}$ and $H \in \mathfrak{h}$ such that aH is generic in $\mathfrak{h}$. Then we can define a non-degenerate closed two-form $\omega^{H,a}$ on LG/T via $\omega^{H,a} = \mathrm{pr}_1^*\omega^a + \mathrm{pr}_2^*\omega_0^{aH}$, where we have identified LG/T with $LG/G \times G/T$ as before, and $\omega_{eT}^a(X,Y) = \int_0^1 \langle aX'(t), y(t)\rangle\,\mathrm{d}t$. Furthermore, let us choose $b \in \mathbb{R}_{>0}$ and $K \in \mathfrak{h}$. Then we can define an $\mathbb{R}$-action on LG/T via

$$u : \gamma \mapsto \exp(buK) R_{bu}(\gamma) \exp(-buK).$$

If bK is generic, the fixed point set of this action is $Q^\vee \times W$ as before. The vector field defined by this $\mathbb{R}$-action at a point $\gamma T \in LG/T$ is $b\gamma' + \mathrm{ad}(bK)(\gamma)$, and as in Lemma 3.6 one can deduce that this vector field is exactly the Hamiltonian vector field on LG/T corresponding to the function $J_{H,K,a,b}$, where

$$J_{H,K,a,b}(\gamma) = \frac{ab}{2}\int_0^1 \|\gamma'(t)\gamma^{-1}(t) + \gamma(t)K\gamma^{-1}(t) - H\|^2\,\mathrm{d}t.$$

The skew symmetric automorphism relating the Riemannian and the new symplectic structure on LG/T is now $B_\sigma(eT) = a(\partial/\partial t) + \mathrm{ad}(aH)$ and its zeta-regularized Pfaffian is given by

$$\mathrm{Pf}_\zeta(B_\sigma)(eT) = a^{\dim \mathfrak{g}}(2\pi)^{\dim \mathfrak{g}} \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H)).$$

Accordingly, we get

$$Z_{eT}(K) = b^{\dim \mathfrak{g}}(2\pi)^{\dim \mathfrak{g}}(\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(K)).$$

Set $c = 1/ab$. Then the same calculation as the one leading to Theorem 3.7 gives

$$\sum_{w\in W}\sum_{\beta\in Q^\vee} (-1)^{l(w)}\,\mathrm{e}^{-(1/2c)\|\beta+w(K)-H\|^2} = \frac{\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(-H))2\sin(\pi\alpha(K))}{(\sqrt{2\pi})^l\,\mathrm{vol}(Q^\vee)}$$
$$\times \sum_{\lambda\in P_{++}} |\chi_\lambda(\exp(-H))\chi_\lambda(\exp(K))\,\mathrm{e}^{-(c/2)\|\lambda+\rho\|^2}.$$

So putting everything together gives the following generalization of Theorem 3.7.

Theorem 3.9. *Let a, b, K, H as above. Then the following identity is valid*:

$$\int_{LG/T} \mathrm{e}^{-J_{H,K,a,b}}\,\mathrm{d}\sigma = \frac{1}{(ab)^{\dim\mathfrak{g}}(2\pi)^{l+\dim\mathfrak{g}}\,\mathrm{vol}(Q^\vee)} \times \sum_{\lambda\in P_+} \chi_\lambda(\exp(-H))\chi_\lambda(\exp(K))\,\mathrm{e}^{-(1/2ab)\|\lambda+\rho\|^2}.$$

In [7], Frenkel has shown how for certain H and b, the numerator of the right-hand side of the equation in Theorem 3.9 can be interpreted as the numerator of the Kac–Weyl character formula for highest weight representations of the untwisted affine Lie algebra corresponding to $\mathfrak{g}$ evaluated at K, b. So Theorem 3.9 gives a realization of the affine characters as integrals over a coadjoint orbit. This is one of the main features of Kirillov's "method of orbits" in the representation theory of Lie groups [14]. In the next paragraph, we will compare our approach to these orbital integrals via the Liouville functional to the analytic approach using the Wiener measure on a compact Lie group developed in [7].

3.4. Comparison with Wiener measure

In his heuristic deduction of the index theorem for the Dirac operator on a Riemannian manifold, Witten (cf. [1]) has suggested that the Wiener measure on a Riemannian manifold M should be closely connected to the "Riemannian measure" on the loop space of M. (Of course, the loop space of M is not a symplectic manifold in our sense, but one can extend the definition of the "Riemannian volume form" $\mathrm{d}\sigma$ to this case.) In the case of the homogeneous space LG/T we consider, we can make this connection between $\mathrm{d}\sigma$ and the Wiener measure $\mathrm{d}\varpi$ on the compact group G explicit. In fact, one can embed LG/T into a space of continuous maps $[0, 1] \to G$ on which the Wiener measure is defined. So the first guess would be that after possibly some identifications one has $\mathrm{d}\sigma = \mathrm{d}\varpi$. But by construction, $\mathrm{d}\sigma$ is invariant under left translations, whereas the Wiener measure is only quasi-invariant: Set

$$C_G = \{z : [0,1] \to G | z(0) = e, z \text{ continuous}\}$$

and let $f : C_G \to \mathbb{R}$ be integrable with respect to the Wiener measure ϖ on C_G. Then

$$\int_{C_G} f(z)\,\mathrm{d}\varpi(z) = \int_{C_G} f(gz)\,\mathrm{e}^{-\langle z'z^{-1}, g^{-1}g'\rangle-(1/2)\langle g'g^{-1}, g'g^{-1}\rangle}\,\mathrm{d}\varpi(z),$$

where $g \in C_G$ and $\langle X, Y\rangle = \int_0^1 \langle X(t), Y(t)\rangle\,\mathrm{d}t$ for $X, Y \in C([0, 1], \mathfrak{g})$. See [7] for more details. To get rid of this defect, we will replace $\mathrm{d}\varpi(z)$ with $\mathrm{d}\tilde{\varpi} = \mathrm{e}^{(1/2)\|z'z^{-1}\|^2}\,\mathrm{d}\varpi(z)$. The new "measure" $\mathrm{d}\tilde{\varpi}$ is indeed invariant under left translations and we will formally have $\mathrm{d}\sigma = \mathrm{d}\tilde{\varpi}$ as desired.

To be more concrete, remember the classification of the LG-orbits on $L\mathfrak{g} \times \{1\}$ from Proposition 3.1: Let $\mathcal{O}_g$ denote the conjugacy class of G containing the element g and set

$$C_{G,\mathcal{O}_g} = \{z \in C_G \text{ such that } z(1) \in \mathcal{O}_g\}.$$

Let us identify $L\mathfrak{g} \times \{1\}$ with $L\mathfrak{g}$. Then LG acts via $\gamma : X \mapsto \gamma X\gamma^{-1} + \gamma'\gamma^{-1}$. After identifying LG/T with the LG-orbit through H, we can define a map

$$\phi : LG/T \to C_{G,\mathcal{O}_{\exp(H)}}$$

via

$$\gamma H\gamma^{-1} + \gamma'\gamma^{-1} \mapsto z_{\gamma H\gamma^{-1}+\gamma'\gamma^{-1}},$$

where z_X denotes the fundamental solution of the differential equation $z' = -Xz$. Now one can identify C_G with a subspace $(L\mathfrak{g})^*_0$ of $(L\mathfrak{g})^*$ (see [7]), and in this identification, $C_{G,\mathcal{O}_{\exp(H)}}$ can be viewed as the closure in $(L\mathfrak{g})^*_0$ of the coadjoint orbit containing H.

The most natural measure $C_{G,\mathcal{O}_g}$ is the conditional Wiener measure $\varpi_{G,\mathcal{O}_g}$ constructed in [7]. Let $f : C_{G,\mathcal{O}_g} \to \mathbb{R}$ be an integrable function with respect to this measure. The integral over f will be denoted by

$$\int_{C_{G,\mathcal{O}_g}} f(z)\,\mathrm{d}\varpi_{G,\mathcal{O}_g}(z).$$

This integral has the quasi-invariance properties stated above. As outlined before, let us replace $\mathrm{d}\varpi_{G,\mathcal{O}_g}(z)$ with $\mathrm{d}\tilde{\varpi}_{G,\mathcal{O}_g}(z) = \mathrm{e}^{(1/2)\|z'z^{-1}\|^2}\,\mathrm{d}\varpi_{G,\mathcal{O}_g}(z)$ such that we get

$$\int_{C_{G,\mathcal{O}_g}} f(\gamma z)\,\mathrm{d}\tilde{\varpi}_{G,\mathcal{O}_{\gamma(2\pi)g}}(z) = \int_{C_{G,\mathcal{O}_g}} f(z)\,\mathrm{d}\tilde{\varpi}_{G,\mathcal{O}_g}(z)$$

for all $\gamma \in C_G$.

Now let us define a function $\tilde{J}H : C_{G,\mathcal{O}_{\exp(H)}} \to \mathbb{R}$ via $\tilde{J}_H(z) = \frac{1}{2}\|z'z^{-1} + H\|^2$. One checks directly that $\phi^*\tilde{J}_H = J_H$ with $\phi : LG/T \to C_{G,\mathcal{O}_{\exp(H)}}$ as before. The main result of this section is the following proposition.

Proposition 3.10.

$$\int_{LG/T} \mathrm{e}^{-J_{H(\gamma)}}\,\mathrm{d}\sigma(\gamma) = c\int_{C_{G,\mathcal{O}_{\exp(H)}}} \mathrm{e}^{-\tilde{J}_H(z)}\mathrm{d}\tilde{\varpi}_{G,\mathcal{O}_{\exp(H)}}(z)$$

with $c = \mathrm{e}^{(1/2)\|\rho\|^2}(2\pi)^{l+\dim\mathfrak{g}}\,\mathrm{vol}(Q^\vee)$.

So up to a constant which does not depend on H, the Wiener measure on $C_{G,\mathcal{O}_{\exp(H)}}$ and the "Riemannian measure" on LG/T are equal.

Proof. For $z \in C_{G,\mathcal{O}_{\exp(H)}}$ we have

$$\tilde{J}_H(z) = \tfrac{1}{2}\|z'z^{-1}\|^2 + \langle H, z'z^{-1}\rangle + \tfrac{1}{2}\|H\|^2,$$

so that we get

$$\int_{C_{G,\mathcal{O}_{\exp(H)}}} \mathrm{e}^{-\tilde{J}_H(z)}\,\mathrm{d}\tilde{\varpi}_{C_{G,\mathcal{O}_{\exp(H)}}}(z) = \mathrm{e}^{-(1/2)\|H\|^2}\int_{C_{G,\mathcal{O}_{\exp(H)}}} \mathrm{e}^{-\langle H,z'z^{-1}\rangle}\,\mathrm{d}\varpi_{C_{G,\mathcal{O}_{\exp(H)}}}(z).$$

The last integral was computed in [7, Theorem 5.2.15].

$$\mathrm{e}^{-(1/2)\|H\|^2}\int_{C_{G,\mathcal{O}_{\exp(H)}}}\mathrm{e}^{-\langle H,z'z^{-1}\rangle}\,\mathrm{d}\varpi_{G,\mathcal{O}_{\exp(H)}}(z)=\sum_{\lambda\in P_+}|\chi_\lambda(H)|^2\,\mathrm{e}^{-(1/2)\|\lambda+\rho\|^2-\|\rho\|^2}.$$

Comparing this result with Theorem 3.7 finishes the proof. □

Remark 3.11. Theorem 3.10 can be easily extended to the function $J_{H,K,a,b}$ which shows that the approaches to the orbit theory of affine Lie algebras via the Wiener measure and via the Liouville functional are equivalent. Of course, following Witten's assumption that our "Riemannian volume form" does indeed have something to do with the Wiener measure, this is the result one should have expected.

3.5. The twisted partition function

In this section, we will "integrate" functions on the coadjoint orbits of twisted loop groups. In this case, the calculation of the zeta-regularized Pfaffian gives a "duality" between root systems which also appears in the calculation of characters of certain non-connected compact Lie groups (cf. [27]).

Let ψ be an outer automorphism of order $\mathrm{ord}(\psi)=r$ of the simply connected compact semi-simple Lie group G such that ψ acts as an automorphism of the Dynkin diagram on the root system of the complexified Lie algebra $\mathfrak{g}\otimes\mathbb{C}$. Let us denote by $L(G,\psi)$ the corresponding twisted loop group:

$$L(G,\psi)=\{\gamma\in LG|\psi(\gamma(t))=\gamma(t+1/r)\,\text{for all}\,t\in[0,1]\}.$$

The Lie algebra of $L(G,\psi)$ will be denoted by $L(\mathfrak{g},\psi)$. By restriction, the symmetric invariant form $\langle\cdot,\cdot\rangle$ and the antisymmetric form ω on $L\mathfrak{g}$ give a symmetric and an anti-symmetric form on $L(\mathfrak{g},\psi)$, which will be denoted by the same symbols. The form $\langle\cdot,\cdot\rangle$ is non-degenerate on $L(\mathfrak{g},\psi)$ and defines a Riemannian structure σ on $L(G,\psi)$ by left translation. The form ω is degenerate exactly in the subspace of constant loops so that it defines a symplectic form on $L(G,\psi)/G^\psi$, where G^ψ denotes the group of fixed points under the automorphism ψ. Since we chose G to be compact and semi-simple, so will be G^ψ with maximal torus T^ψ. The manifold G^ψ/T^ψ can be viewed as a coadjoint orbit of G^ψ through a generic $H\in\mathfrak{h}^\psi$. As before, the Kirillov form on such orbit will be denoted by ω_0^H. After identifying $L(G,\psi)/T^\psi$ with $L(G,\psi)/G^\psi\times G^\psi/T^\psi$, we can define a symplectic structure ω^H on $L(G,\psi)/T^\psi$ via $\omega^H=\mathrm{pr}_1^*\omega+\mathrm{pr}_2^*\omega_0^H$. As in Section 3.1, the skew symmetric endomorphism of the tangent space at eT^ψ of $L(G,\psi)/T^\psi$ relating the Riemannian metric σ and the symplectic form is given by $B_{\sigma,eT^\psi}: X\mapsto X'+\mathrm{ad}H(X)$. The calculation of the zeta-regularized Pfaffian of B_{σ,eT^ψ} is essentially the same as the calculation of the zeta-regularized Pfaffian in Section 3.1 but we have to be more careful with the multiplicities of the eigenvalues.

Let Δ denote the root system of $\mathfrak{g}\otimes\mathbb{C}$ and let Δ^ψ denote the "folded" root system, i.e. $\Delta^\psi=\{\bar\alpha|\alpha\in\Delta\}$, where $\bar\alpha$ denotes the element $\bar\alpha=(1/\mathrm{ord}(\psi))\sum_{i=1}^{\mathrm{ord}(\psi)}\psi^i(\alpha)$. Let us assume for the moment that Δ is an irreducible root system of type ADE but not of type A_{2n}. In this case Δ^ψ is a root system of type BCFG. Let Δ_s^ψ and Δ_l^ψ denote the subsets of short and long roots in Δ^{ψ}, respectively. Now we can use [12], Proposition 6.3 to see that the eigenvalues of $B_{\sigma,eT^{\psi}}$ are given by

$$\{\pm 2\pi\mathrm{i}n | n\in\mathbb{N}_{>0}\}\cup\{2\pi\mathrm{i}(\pm\alpha(H)+n)|\alpha\in\Delta_s^{\psi}, n\in\mathbb{Z}\}\\ \cup\{2\pi\mathrm{i}(\pm\alpha(H)+nr)|\alpha\in\Delta_l^{\psi}, n\in\mathbb{Z}\}.$$

Furthermore, if Δ is of type X_N in the notation of [12] then for an arbitrary eigenvalue $2\pi\mathrm{i}\lambda$ of $B_{\sigma,eT^{\psi}}$, we have $\mathrm{mult}(2\pi\mathrm{i}\lambda)=1$ if $\lambda=\alpha(H)+n$ with $\alpha\in\Delta^{\psi}$. In case $\lambda=n$ we have $\mathrm{mult}(2\pi\mathrm{i}\lambda)=l=\dim(T^{\psi})$ if r divides n and $\mathrm{mult}(2\pi\mathrm{i}\lambda)=(N-l)/(r-1)$ if $r\nmid n$. So the zeta-regularized determinant of $B_{\sigma,eT^{\psi}}$ is given by

$$\begin{aligned}\det{}_{\zeta}(B_{\sigma,eT^{\psi}}) &= \prod_{\alpha\in\Delta_{s+}^{\psi}}\left((2\pi\alpha(H))^2\prod_{n=1}^{\infty}((2\pi)^2(n^2-\alpha(H)^2))^2\right)_{\zeta}\\ &\quad\times\prod_{\alpha\in\Delta_{l+}^{\psi}}\left((2\pi\alpha(H))^2\prod_{n=1}^{\infty}((2\pi)^2(r^2n^2-\alpha(H)^2))^2\right)_{\zeta}\\ &\quad\times\left(\prod_{n=1}^{\infty}(2\pi n)^2\right)_{\zeta}^{(N-l)/(r-1)}\left(\prod_{n=1}^{\infty}r^2(2\pi n)^2\right)_{\zeta}^{l-((N-l)/(r-1))}\\ &= \prod_{\alpha\in\Delta_{s+}^{\psi}}4\sin^2(\pi\alpha(H))\prod_{\alpha\in\Delta_{l+}^{\psi}}4\sin^2\left(\frac{\pi}{r}\alpha(H)\right)\\ &\quad\times\left(\prod_{n=1}^{\infty}2\pi n\right)_{\zeta}^{(2(N-l)/(r-1))+4|\Delta_{s+}^{\psi}|}\left(\prod_{n=1}^{\infty}2\pi rn)\right)_{\zeta}^{2l-(2(N-l)/(r-1))+4|\Delta_{l+}^{\psi}|}\\ &= (2\pi)^{2\dim\mathfrak{g}^{\psi}}r^{l-((N-l)/(r-1))+2|\Delta_{l+}^{\psi}|}\prod_{\alpha\in\Delta_{+}^{\psi\vee}}4\sin^2(\pi\alpha(H)),\end{aligned}$$

where $\Delta^{\psi\vee}$ denotes the root system dual to Δ^{ψ}. So analogous to Lemma 3.3, the zeta-regularized Pfaffian of $B_{\sigma,eT^{\psi}}$ now reads

$$\mathrm{Pf}_{\zeta}(B_{\sigma})(eT^{\psi})=(2\pi)^{\dim\mathfrak{g}^{\psi}}(\sqrt{r})^{l-((N-l)/(r-1))+2|\Delta_{l+}^{\psi}|}\prod_{\alpha\in\Delta_{+}^{\psi\vee}}2\sin(\pi\alpha(H)).$$

Note that up to a constant coefficient, the Pfaffian $\mathrm{Pf}_{\zeta}(B_{\sigma})(eT^{\psi})$ is exactly the denominator of the Weyl character formula for the compact Lie group with root system $\Delta^{\psi\vee}$, or equivalently, the denominator of the characters on the outer component of the principal extension of the Lie group with root system Δ (cf. [27]). In any case, we see that the symplectic form and the Riemannian metric on $L(G,\psi)/T^{\psi}$ are compatible in the sense of Section 2.1.

The $S^1\times T$-action on LG/T considered in Section 3.2 defines an $S^1\times T^{\psi}$-action on $L(G,\psi)/T^{\psi}$ by restriction. That is, $S^1\times T^{\psi}$ acts on $L(G,\psi)/G^{\psi}$ by “twisted rotation” and T^{ψ} acts on G^{ψ}/T^{ψ} by conjugation. A similar calculation to the corresponding one for the untwisted case shows that the fixed point set of the "twisted rotation action" of $S^1 \times T^\psi$ on $L(G,\psi)/G^\psi$ is given by the lattice $M \subset \mathfrak{h}$ which is generated by the long roots in Δ^ψ (where we have identified $\mathfrak{h}$ with $\mathfrak{h}^*$ via the negative of the Killing form on G). The fixed point set of the T^ψ-action on G^ψ/T^ψ is given by the Weyl group W^ψ of G^ψ. As a set, $W^\psi \times M$ can be identified with the affine Weyl group belonging to the twisted affine Lie algebra corresponding to $L(G,\psi)$ (see e.g. [12] or [27] for more on the theory of twisted affine Lie algebras). The number $\#(w,\beta)$ for $(w,\beta) \in W^\psi \times M$ is again given by $\#(w,\beta) = l((w,\beta))$, where $l((w,\beta))$ denotes the length of (w,β) in $W^\psi \times M$.

Let us compute $Z_{eT^\psi}(H)$: A calculation similar to the one we used for the Pfaffian now gives

$$Z_{eT^\psi}(H) = c \prod_{\alpha \in \Delta_+^{\psi\vee}} 2\sin(\pi\alpha(H))$$

with $c = (\sqrt{2\pi})^l(\sqrt{r})^{l-((N-l)/(r-1))+2|\Delta_{l+}^\psi|}$.

Now let us consider the function $J_H|L(G,\psi)/T^\psi : L(G,\psi)/T^\psi \to \mathbb{R}$, where J_H is the function we considered in Section 3.3. That is

$$J_H(\gamma) = \frac{1}{2}\int_0^1 \|\gamma'(t)\gamma^{-1}(t) + \gamma(t)H\gamma^{-1}(t) - H\|\,\mathrm{d}t.$$

Since for generic $H \in \mathfrak{h}^\psi$, the corresponding $\mathbb{R}$-action on $L(G,\psi)/T^\psi$ is just the restriction of the corresponding $\mathbb{R}$-action on LG/T, it follows from Lemma 3.6 that the Hamiltonian vector field on $L(G,\psi)/T^\psi$ corresponding to J_H is the vector field generated by the $\mathbb{R}$-action. Therefore, we can calculate the Liouville functional of J_H: A calculation in [27, Section 4.2] implies

$$\sum_{w\in W^\psi}\sum_{\beta\in M}(-1)^{l(w)}\mathrm{e}^{-(1/2)\|\beta+w(H)-H\|^2} = \frac{\prod_{\alpha\in\Delta_+^1} 4\sin^2(\pi\alpha(H))}{(2\pi)^{1/2}\,\mathrm{vol}(M)} \sum_{\lambda\in P_+(\Delta^1)} |\chi_\lambda(\exp(H))|^2\,\mathrm{e}^{-(1/2)\|\lambda+\rho^\psi\|^2},$$

where $P(\Delta^{\psi\vee})$ denotes the weight lattice of the root system Δ^1, $P_+(\Delta^{\psi\vee})$ denotes the cone of dominant weights, χ_λ denotes the irreducible character of the compact simply connected semi-simple Lie group of the same type as the root system $\Delta^{\psi\vee}$, and ρ^ψ is the half sum of the positive roots of $\Delta^{\psi\vee}$. Putting this together with the calculation of the Pfaffian gives the following proposition.

Proposition 3.12.

$$\int_{L(G,\psi)/T^\psi} \mathrm{e}^{-J_H(\gamma)}\,\mathrm{d}\sigma(\gamma) = C \sum_{\lambda\in P_+(\Delta^{\psi\vee})} |\chi_\lambda(\exp(H))|^2\,\mathrm{e}^{-(1/2)\|\lambda+\rho^\psi\|^2}$$

with $c = ((2\pi)^{l+\dim\mathfrak{g}^\psi}\,\mathrm{vol}(M) r^{l-((N-l)/(r-1))+2|\Delta_{l+}^\psi|})^{-1}$.

Remark 3.13. If Δ is of type A_{2n}, the root system Δ^{ψ} is a non-reduced root system of type BC_n. That is, three different root lengths occur. In this case, a similar calculation shows that after replacing the root system $\Delta^{\psi\vee}$ with a reduced root system of type C_n, Proposition 3.12 still holds true.

As in the non-twisted case, one can compare the Liouville functional with a certain Wiener measure: The space $L(G, \psi)/T^{\psi}$ can be embedded into space of paths in the outer component $G\psi$ of the non-connected Lie group $G \rtimes \langle\psi\rangle$ on which the Wiener measure is defined (see [27] for details). Comparison of the Wiener measure on this path space with the "Riemannian measure" on $L(G, \psi)/T^{\psi}$ yields the same result as Section 3.4. The calculations leading to Theorem 3.9 can be easily adjusted to the twisted case so that we have an analogous re-interpretation of the orbital theory for the twisted loop groups developed in [27].

4. The WZW model

4.1. The WZW model at level κ

The WZW model is a quantum field theory on a Riemann surface Σ with values in a simply connected semi-simple compact Lie group G (or more generally in its complexification $G_{\mathbb{C}}$). See e.g. [9] for an introduction to quantum field theory in general and the WZW model on an arbitrary Riemann surface in particular. We will only be interested in the case when Σ is the elliptic curve Σ_τ, i.e. the torus $S^1 \times S^1 + \mathbb{R}^2/\mathbb{Z}^2$ together with a complex structure which is defined by declaring $f : \Sigma_\tau \to \mathbb{C}$ to be holomorphic if $\bar{\partial} f := (\partial_s + \tau\partial_t) f = 0$. Here $\tau = \tau_1 + \mathrm{i}\tau_2$ with $\tau_1, \tau_2 \in \mathbb{R}$, $\tau_2 > 0$ denotes the modular parameter of the elliptic curve Σ_τ.

As before, let $\langle\cdot,\cdot\rangle$ denote the Killing form on $\mathfrak{g}_{\mathbb{C}}$ normalized so that the long roots have square length 2 and set $\partial = \partial_s + \bar{\tau}\partial_t$. In our normalization, the action functional of the WZW model at level κ is given by

$$S_{G,\kappa}(g) = -\frac{\kappa\pi}{2\tau_2}\int_{\Sigma} \langle g^{-1}\partial g, g^{-1}\bar{\partial} g\rangle \,\mathrm{d}s\,\mathrm{d}t + \frac{\mathrm{i}\kappa\pi}{3}\int_{B} \mathrm{tr}(\tilde{g}^{-1}\,\mathrm{d}\tilde{g})^{\wedge 3},$$

where B is a three-dimensional manifold with boundary $\partial B = \Sigma$, and $\tilde{g} : B \to G$ is a map such that $\tilde{g}|_{\partial B} = g$ and tr denotes the negative of the normalized Killing form as well. The second term in the action is the so-called Wess–Zumino term. Up to the factor $\mathrm{i}\kappa$, it is the integral over the pull back of the generator of $H^3(G, \mathbb{Z})$ to B via the map $\tilde{g}$. The action $S_{G,k}$ was first studied by Witten [28].

If $\tilde{g}_1$ and $\tilde{g}_2$ are two different extensions of g they differ by a map $\tilde{h} : B \to G$ such that $\tilde{h}|_{\partial B} = e$. But for such $\tilde{h}$ we have $(\pi/3)\int_B \mathrm{tr}(\tilde{h}^{-1}\tilde{h})^{\wedge 3} \in 2\pi\mathbb{Z}$ such that the action $\mathrm{e}^{S_{G,\kappa}(g)}$ is well defined for $\kappa \in \mathbb{Z}$.

Some of the transformation properties of the WZW action are given by the Polyakov–Wiegmann formula (cf. [10,22]).

Proposition 4.1. *Let $g, h : \Sigma \to G$. Then the following identity is valid*:

$$S_{G,\kappa}(gh) = S_{G,\kappa}(g) + S_{G,\kappa}(h) - \frac{\kappa\pi}{\tau_2}\int_{\Sigma} \langle g^{-1}\partial g, \bar{\partial} h h^{-1}\rangle \,\mathrm{d}s\,\mathrm{d}t.$$

Note that the imaginary part of the term $(\kappa\pi/\tau_2)\int_\Sigma\langle g^{-1}\partial g, \bar{\partial}hh^{-1}\rangle\,\mathrm{d}s\,\mathrm{d}t$ is exactly the cocycle in the explicit construction of the central extension $\hat{G}$ of the loop group LG as a quotient (see [8,18]).

Proof. One directly checks that

$$\begin{aligned}
&-\frac{\kappa\pi}{2\tau_2}\int_\Sigma\langle h^{-1}g^{-1}\partial(gh), h^{-1}g^{-1}\bar{\partial}(gh)\rangle\,\mathrm{d}s\,\mathrm{d}t\\
&\quad=-\frac{\kappa\pi}{2\tau_2}\int_\Sigma\langle g^{-1}\partial g, g^{-1}\bar{\partial}g\rangle\,\mathrm{d}s\,\mathrm{d}t-\frac{\kappa\pi}{2\tau_2}\int_\Sigma\langle h^{-1}\partial h, h^{-1}\bar{\partial}h\rangle\,\mathrm{d}s\,\mathrm{d}t\\
&\qquad-\frac{\kappa\pi}{\tau_2}\int_\Sigma(\langle g^{-1}\partial_s g, \partial_s hh^{-1}\rangle+\tau_1\langle g^{-1}\partial_s g, \partial_t hh^{-1}\rangle+\tau_1\langle g^{-1}\partial_t g, \partial_s hh^{-1}\rangle\\
&\qquad+\tau\bar{\tau}\langle g^{-1}\partial_t g, \partial_t hh^{-1}\rangle)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}((gh)^{-1}\,\mathrm{d}(gh))^{\wedge 3}\\
&\quad=\frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}(g^{-1}\,\mathrm{d}g)^{\wedge 3}+\frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}(h^{-1}\,\mathrm{d}h)^{\wedge 3}\\
&\qquad+\mathrm{i}\kappa\pi\int_B \mathrm{tr}((g^{-1}\,\mathrm{d}g)^{\wedge 2}\wedge \mathrm{d}hh^{-1}+g^{-1}\,\mathrm{d}g\wedge(\mathrm{d}hh^{-1})^{\wedge 2})\\
&\quad=\frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}(g^{-1}\,\mathrm{d}g)^{\wedge 3}+\frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}(h^{-1}\,\mathrm{d}h)^{\wedge 3}+\mathrm{i}\kappa\pi\int_B \mathrm{dtr}(g^{-1}\,\mathrm{d}g\wedge \mathrm{d}hh^{-1}).
\end{aligned}$$

Now Stoke's theorem implies $\mathrm{i}\kappa\pi\int_B\mathrm{dtr}(g^{-1}\,\mathrm{d}g\wedge\mathrm{d}hh^{-1})=\mathrm{i}\kappa\pi\sum_\Sigma\mathrm{tr}(g^{-1}\,\mathrm{d}g\wedge\mathrm{d}hh^{-1})$. Writing $\mathrm{d}g=\partial_s g\,\mathrm{d}s+\partial_t g\,\mathrm{d}t$ and $\mathrm{d}h=\partial_s h\,\mathrm{d}s+\partial_t h\,\mathrm{d}t$ yields the assertion. □

Let $H\in\mathfrak{h}$ be generic. We will extend the WZW action slightly by adding an H-dependent term: Set

$$\begin{aligned}
S_{G,H,\kappa}(g)=S_{G,\kappa}(g)+\frac{\kappa\pi}{\tau_2}\int_{\Sigma_\tau}&(\langle g^{-1}\partial g, H\rangle-\langle\bar{\partial}gg^{-1}, H\rangle-\langle H, g^{-1}Hg\rangle\\
&+\langle H, H\rangle)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

$S_{G,H,\kappa}(g)$ is essentially the action of the gauged WZW model studied in [10]. The partition function of the gauged WZW model at level κ is formally given by the integral

$$\int_{C^\infty(\Sigma_\tau,G_{\mathbb{C}})}\mathrm{e}^{S_{G,H,\kappa}(g)}\mathcal{D}(g),$$

where the integration ranges over all C^∞-maps $g:\Sigma\to G_{\mathbb{C}}$.

The main goal of the next two paragraphs is to make sense of this integral and to calculate the partition function using the Liouville functional approach. To do this, we will have to work in a complex setting as described in Section 2.2.

4.2. Double loop groups and a torus-action

From now on we will consider the group $G_{\mathbb{C}}$, which is the complexification of the compact semi-simple simply connected Lie group G. Let $LLG_{\mathbb{C}}$ denote the set of all C^{∞}-maps from the torus $S^1 \times S^1$ to $G_{\mathbb{C}}$. Together with pointwise multiplication $LLG_{\mathbb{C}}$ becomes a Lie group with Lie algebra $LL\mathfrak{g}_{\mathbb{C}}$, the set of C^{∞}-maps from $S^1 \times S^1$ to the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$. A one-dimensional central extension $\widetilde{LLG}_{\mathbb{C}}$ and the corresponding coadjoint representation of $LLG_{\mathbb{C}}$ was constructed in [5]. One of the results of [5] is that a generic coadjoint orbit of $\widetilde{LLG}_{\mathbb{C}}$ is isomorphic to $LLG_{\mathbb{C}}/T_{\mathbb{C}}$, where T is a maximal torus in G and $T_{\mathbb{C}}$ denotes its complexification.

Let $H \in \mathfrak{h}$ be generic and choose a modular parameter $\tau = \tau_1 + \mathrm{i}\tau_2 \in \mathbb{C}$ such that $\tau_2 > 0$. With these choices made, we can define a non-degenerate closed two-form and an $\mathbb{R}$-action on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ in total analogy with Sections 3.1 and 3.2, respectively: Note that we have $LLG_{\mathbb{C}}/T_{\mathbb{C}} \cong \Omega\Omega G_{\mathbb{C}} \times G_{\mathbb{C}}/T_{\mathbb{C}}$, where $\Omega\Omega G_{\mathbb{C}}$ denotes the set of maps $g \in LLG_{\mathbb{C}}$ such that $g(1,1) = e$. Let $X, Y : S^1 \times S^1 \to \mathfrak{g}_{\mathbb{C}}$ be elements of the Lie algebra of $LLG_{\mathbb{C}}$. Then

$$\omega_e(X, Y) = \frac{\pi}{\tau_2} \int_{S^1 \times S^1} \langle \bar{\partial} X(s,t), Y(s,t) \rangle \, \mathrm{d}s \, \mathrm{d}t$$

defines a $\mathbb{C}$-valued skew symmetric bilinear form on $LL\mathfrak{g}_{\mathbb{C}}$ which is degenerate on the set of holomorphic maps. Since Σ_τ is compact, any holomorphic map from Σ_τ to $G_{\mathbb{C}}$ has to be constant. Hence, by left translation, ω defines a non-degenerate $\mathbb{C}$-valued two-form on $LLG_{\mathbb{C}}/G_{\mathbb{C}} \cong \Omega\Omega G_{\mathbb{C}}$. We can choose a $\mathbb{C}$-valued two-form ω_0^H on $G_{\mathbb{C}}/T_{\mathbb{C}}$ which is defined via $\omega^H_{0,eT_{\mathbb{C}}}(A, B) = (\pi/\tau_2)\langle H, [A, B]\rangle$ for $A, B \in T_{eT_{\mathbb{C}}}$ and extended to $G_{\mathbb{C}}/T_{\mathbb{C}}$ via left translation. Putting these two-forms together, we obtain a non-degenerate $\mathbb{C}$-valued two-form $\omega^H = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega_0^H$ on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$. As in the case of loop groups, one checks that ω^H is closed. Hence it can be considered as a $\mathbb{C}$-valued symplectic form in the sense of Section 2.2.

Our next goal is to find an almost complex structure J on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ which is compatible with the complex-valued symplectic form ω^H in the sense of Section 2.2. Consider the decomposition $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} = \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ into one-dimensional root spaces. Here, as before, Δ denotes the root system of $\mathfrak{g}_{\mathbb{C}}$. For each $\alpha \in \Delta$ choose $X_\alpha \in \mathfrak{g}_\alpha$ such that $\langle X_\alpha, X_{-\alpha} \rangle = 1$. Furthermore, choose an orthonormal basis $H_1, \dots, H_l$ of $\mathfrak{h}$. Then any $X \in LL\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ can be written as

$$X(s,t) = \sum_{(n,m)\in\mathbb{Z}^2} \sum_{\alpha\in\Delta} c_{n,m,\alpha} X_\alpha \, \mathrm{e}^{2\pi \mathrm{i}(ns+mt)} + \sum_{\substack{(n,m)\in\mathbb{Z}^2 \\ (n,m)\neq(0,0)}} \sum_{j=1}^{l} c_{n,m,j} H_j \, \mathrm{e}^{2\pi \mathrm{i}(ns+mt)}$$

with $c_{n,m,\alpha}, c_{n,m,j} \in \mathbb{C}$.

As always, let $\tau = \tau_1 + \mathrm{i}\tau_2$ denote the modular parameter of the elliptic curve. Let Δ_+ be a set of positive roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to some basis of Δ. Then let us decompose the set $\tilde{\tilde{\Delta}} = \{(\alpha, n, m) | \alpha \in \Delta \cup \{0\}, (n,m) \in \mathbb{Z}^2 (\alpha, n, m) \neq (0,0,0)\}$ into $\tilde{\tilde{\Delta}}_+ \cup \tilde{\tilde{\Delta}}_-$ via defining (α, n, m) to be positive if either $n + \tau_1 m > 0$ or $n + \tau_1 m = 0$ and $m < 0$ or $n = m = 0$ and $\alpha \in \Delta_+$. Now we can define an $\mathbb{R}$ linear anti-involution J of $LL\mathfrak{g}_{\mathbb{C}}$ which anti-commutes with multiplication by i as follows.

For $c \in \mathbb{C}$, set

$$J(c_{n,m,\alpha} X_\alpha \,\mathrm{e}^{2\pi \mathrm{i}(ns+mt)}) = \begin{cases} \bar{c}_{n,m,\alpha} X_{-\alpha}\,\mathrm{e}^{-2\pi \mathrm{i}(ns+mt)} & \text{if}\,(\alpha, n, m) \in \tilde{\tilde{\Delta}}_+ \\ -\bar{c}_{n,m,\alpha} X_{-\alpha}\,\mathrm{e}^{-2\pi \mathrm{i}(ns+mt)} & \text{if}\,(\alpha, n, m) \in \tilde{\tilde{\Delta}}_-. \end{cases}$$

Analogously, set

$$J(c_{n,m,\neq} H_\nu \,\mathrm{e}^{2\pi \mathrm{i}(ns+mt)}) = \begin{cases} \bar{c}_{n,m,\nu} H_\nu\,\mathrm{e}^{-2\pi \mathrm{i}(ns+mt)} & \text{if}\,(0, n, m) \in \tilde{\tilde{\Delta}}_+ \\ -\bar{c}_{n,m,\nu} H_\nu\,\mathrm{e}^{-2\pi \mathrm{i}(ns+mt)} & \text{if}\,(0, n, m) \in \tilde{\tilde{\Delta}}_-. \end{cases}$$

Now if the the set of positive roots Δ_+ is chosen in such a way that H lies in the fundamental Weyl chamber with respect to Δ_+, it is straightforward to check that the complex structure J is indeed compatible with the symplectic form ω^H in the sense of Section 2.2. Furthermore, J commutes with the natural $T_\mathbb{C}$-action on $LL\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$. So J defines an automorphism of the tangent bundle and thus an almost complex structure on $LLG_\mathbb{C}/T_\mathbb{C}$ by left translation.

A bilinear form on $LL\mathfrak{g}_\mathbb{C}$ is given by

$$\sigma(X, Y) = \pi \int_{S^1 \times S^1} \langle X(s,t), Y(s,t)\rangle \,\mathrm{d}s\,\mathrm{d}t.$$

Hence the skew-symmetric automorphism of the tangent space relating ω^H and σ reads $B_{\sigma,eT_\mathbb{C}} = (1/\tau_2)(\bar{\partial} + \mathrm{ad}\, H)$. As before, let Δ denote the root system of G. Then the eigenvalues of $B_{\sigma,eT_\mathbb{C}}$ are

$$\frac{2\pi \mathrm{i}}{\tau_2}(n + \tau m + \alpha(H))\,\text{for}\,\alpha \in \Delta,\ n, m \in \mathbb{Z},$$
$$\frac{2\pi \mathrm{i}}{\tau_2}(n + \tau m)\,\text{for}\, n, m \in \mathbb{Z}\,\text{and}\, n \neq 0\,\text{or}\, m \neq 0.$$

The multiplicity of the eigenvalues in the first series is 1 and the multiplicity of the eigenvalues in the second series is $l = \dim_\mathbb{R} \mathfrak{h}$. This can be seen using the root space decomposition of the semi-simple Lie algebra $\mathfrak{g}_\mathbb{C}$.

Unfortunately, we cannot calculate the zeta-regularized product of the eigenvalues of $B_{\sigma,eT_\mathbb{C}}$ since the product ranges over a series of complex numbers. To avoid this difficulty, we will introduce an appropriate torus-action on $LLG_\mathbb{C}/T_\mathbb{C}$ and then calculate the product $\mathrm{Pf}_\zeta\, B_{\sigma,eT_\mathbb{C}} Z_{eT_\mathbb{C}}(H)$ which will be the denominator of the partition function. Here, $Z_{eT_\mathbb{C}}(H)$ is the zeta-regularized product of the "positive" eigenvalues of the torus-action.

There is a natural action of the torus $S^1 \times S^1 \times T$ on $LLG_\mathbb{C}/T_\mathbb{C}$, where the first two factors act by rotating the loops and the second factor acts by conjugation. This action is defined analogously to the $S^1 \times T$ action on LG/T considered in Section 3.2. The fixed point set of this action is the set $Q^\vee \times Q^\vee \times W$, where, as before, $Q^\vee \times Q^\vee$ denotes the lattice of homomorphisms $S^1 \times S^1 \to T$ and W is the Weyl group of G. This follows from exactly the same calculation which gave the fixed point set of the $S^1 \times T$-action on LG/T in Section 3.2. Furthermore, one checks directly that the $S^1 \times S^1 \times T$-action on the tangent spaces $T_{(\beta_1,\beta_2,w)} LLG_\mathbb{C}/T_\mathbb{C}$ commutes with the complex structure J for all $(\beta_1, \beta_2, w) \in Q^\vee \times Q^\vee \times W$.

The differential $\partial = \partial_s + \bar{\tau}\partial_t$ defines an element in the complexified Lie algebra of $S^1 \times S^1$. So for $H \in \mathfrak{h}$, the pair (∂, H) can be viewed as an element in the complexified Lie algebra $\mathrm{Lie}_{\mathbb{C}}(S^1 \times S^1 \times T)$ of $S^1 \times S^1 \times T$ and hence defines a vector field on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$. The eigenvalues of the corresponding action of (∂, H) on the tangent space at $T_e T_{\mathbb{C}} LLG_{\mathbb{C}}/T_{\mathbb{C}}$ are

$$2\pi\mathrm{i}(n + \bar{\tau}m + \alpha(H)) \text{ for } \alpha \in \Delta,\ n, m \in \mathbb{Z},$$

$$2\pi\mathrm{i}(n + \bar{\tau}m) \text{ for } n, m \in \mathbb{Z} \text{ and } n \neq 0 \text{ or } m \neq 0.$$

Again, the multiplicity of the eigenvalues is 1 if $\alpha \neq 0$ and l if $\alpha = 0$.

Let us compute the product $\mathrm{Pf}_\zeta(B_\sigma)(eT_{\mathbb{C}})Z(H)$. For simplicity, we will compute $(\mathrm{Pf}_\zeta(B_\sigma)(eT_{\mathbb{C}})Z(H))^2$. Remember that in the definition of $Z(H)$, the eigenvalues of the infinitesimal $\mathbb{R}$-action on the tangent spaces at the fixed points of the torus-action have to be divided by 2π, so that we get

$$(\mathrm{Pf}_\zeta(B_\sigma)(eT_{\mathbb{C}})Z(H))^2 = \prod_{\alpha\in\Delta_+}\left(\prod_{(n,m)\in\mathbb{Z}^2}\left(\frac{2\pi}{\tau_2}|n+\tau m+\alpha(H)|^2\right)\right)_\zeta \times\left(\prod_{\substack{(n,m)\in\mathbb{Z}^2\\(n,m)\neq(0,0)}}\left(\frac{2\pi}{\tau_2}|n+\tau m|^2\right)^l\right)_\zeta .$$

Since $H \in \mathfrak{h}$ was chosen to be generic, we have $\alpha(H) \in \mathbb{R}\backslash\mathbb{Z}$ for all $\alpha \in \Delta$. Therefore, we can calculate the zeta-regularized product using the Epstein zeta-functions $\zeta_\tau(s; v)$ and $\tilde{\zeta}_\tau(s; v)$ which are defined in Appendix A. Lemma A.2 implies

$$(\mathrm{Pf}_\zeta(B_\sigma)(eT_{\mathbb{C}})Z(H))^2 = \prod_{\alpha\in\Delta_+}(2\pi)^{2\zeta_\tau(0;\alpha(H))}\left(\prod_{(n,m)\in\mathbb{Z}^2}\left(\frac{1}{\tau_2}|n+\tau m+\alpha(H)|^2\right)^2\right)_\zeta \times(2\pi)^{l\zeta_\tau(0;0)}\left(\prod_{\substack{(n,m)\in\mathbb{Z}^2\\(n,m)\neq(0,0)}}\left(\frac{1}{\tau_2}|n+\tau m|^2\right)\right)^l_\zeta .$$

Now we can use Lemma A.7 to obtain one of the main results of this section.

Proposition 4.2. *The following identity is valid*:

$$\mathrm{Pf}_\zeta(B_\sigma)Z(H) = C(2\pi\sqrt{\tau_2}|\eta(\tau)|^2)^l\prod_{\alpha\in\Delta_+}\left|q^{1/12}\left(e\left(\frac{1}{2}\alpha(H)\right)-e\left(-\frac{1}{2}\alpha(H)\right)\right)\right. \times\prod_{n=1}^{\infty}(1-q^n e(\alpha(H)))(1-q^n e(-\alpha(H)))\Big|^2$$

with $C = (2\pi)^{-1/2}$.

Note, that up to a coefficient, the right-hand side of the equation in Proposition 4.2 is exactly the squared absolute value of the denominator of the Kac–Weyl character formula (cf. [12]).

Remark 4.3. Let us consider the differential operator $(\partial + \mathrm{ad}\, H)(\bar{\partial} + \mathrm{ad} H)$ acting on the space $C^\infty(S^1 \times S^1, \mathfrak{g})$. This operator can be viewed as a non-Abelian generalization of the Laplacian acting on the space $C^\infty(S^1 \times S^1, \mathfrak{h})$ considered in [25]. Up to a coefficient, the product $(\mathrm{Pf}_\zeta(B_\sigma)Z(H))^2$ is the zeta-regularized determinant $\det_\zeta((\partial + \mathrm{ad}\, H)(\bar{\partial} + \mathrm{ad}\, H))$. In the case of finite-dimensional Gaussian integrals, we have the equality

$$\int_{\mathbb{R}^n} \mathrm{e}^{-\langle x, Bx\rangle}\, \mathrm{d}^n x = \frac{1}{\sqrt{\det(B)}},$$

where B denotes a symmetric matrix. Thus, by analogy, the product $(\mathrm{Pf}_\zeta(B_\sigma)Z(H))^{-1}$ can be viewed as the Gaussian integral

$$\int_{C^\infty(S^1 \times S^1, \mathfrak{g})/\mathfrak{h}} \mathrm{e}^{-\langle X, (\partial + \mathrm{ad}\, H)(\bar{\partial} + \mathrm{ad}\, H)X\rangle} \mathcal{D}(X).$$

Maybe more interestingly, each $\alpha(H)$ defines unitary representation of the lattice $\mathbb{Z} + \tau\mathbb{Z}$ via $(m + \tau n) \mapsto \mathrm{e}^{2\pi \mathrm{i}(\alpha(H)m+n)}$. Such representation gives rise to a complex line bundle over the elliptic curve Σ_τ and our calculation of the zeta-regularized determinant of $(\partial + \mathrm{ad} H)(\bar{\partial} + \mathrm{ad}\, H)$ is exactly the calculation leading to the analytic torsion of this line bundle (see [23]).

Finally, we have to calculate the sign $(-1)^{\# p}$ for the fixed points p of the torus-action on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$. We can proceed as in Section 3.2: Let $(\beta_1, \beta_2, w) \in Q^\vee \times Q \vee \times W$ be a fixed point of the $S^1 \times S^1 \times T$-action on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$. If we choose a representative $g_w \in G$ for each $w \in W$, we can view (β_1, β_2, g_w) as an element of $LLG_{\mathbb{C}}$. Since $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ acts transitively on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$, we can use left translation by (β_1, β_2, g_w) to identify the tangent spaces $T_{eT_{\mathbb{C}}}LLG_{\mathbb{C}}/T_{\mathbb{C}}$ and $T_{(\beta_1,\beta_2,g_w)}LLG_{\mathbb{C}}/T_{\mathbb{C}}$. Since (β_1, β_2, g_w) is an element of the normalizer of $T_{\mathbb{C}}$, this identification is well defined. With this identification, the corresponding infinitesimal $S^1 \times S^1 \times T$-action on $T_{(\beta_1,\beta_2,w)}LLG_{\mathbb{C}}/T_{\mathbb{C}}$ is given by $(\beta_1, \beta_2, g_w)^{-1}(\partial_s + \partial_t + H)(\beta_1, \beta_2, g_w)$ for $(\partial_s + \partial_t + H) \in \mathrm{Lie}(S^1 \times S^1 \times T)$. This defines an action of $Q^\vee \times Q^\vee \times W$ on the set $\tilde{\tilde{\Delta}}$. According to the definition in Section 2.2, $\#(\beta_1, \beta_2, w)$ is the number of elements of $\tilde{\tilde{\Delta}}_+$ which are mapped to $\tilde{\tilde{\Delta}}_-$ under (β_1, β_2, w). Now, as in the case of affine root systems and Weyl groups, one can see (for example by using the fact that the cardinality of the finite root system Δ is even) that we always have $(-1)^{\#(\beta_1,\beta_2,w)} = (-1)^{l(w)}$, where $l(w)$ denotes the length of w in W.

4.3. *Calculation of the WZW partition function*

Now we will calculate the integral defining the partition function of the WZW-model at level κ. Note, that $S_{G,H,\kappa}(g)$ does not depend on the representative of g modulo the complex torus $T_{\mathbb{C}}$. Hence, $S_{G,H,\kappa}$ defines a function on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$. To apply the Liouville functional approach, we have to check that $S_{G,H,\kappa}$ is the Hamiltonian of a vector field on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ which comes from the $S^1 \times S^1 \times T$-action considered in the last paragraph.

Lemma 4.4. *The Hamiltonian vector field on* $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ *corresponding to* $-S_{G,H,1}$ *is exactly the vector field on* $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ *defined by the element* $(\partial, H) \in \mathrm{Lie}_{\mathbb{C}}(S^1 \times S^1 \times T)$ *as in Section* 4.2.

Proof. Let g be a representative of $gT_{\mathbb{C}}$ in $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ and let δg be an infinitesimal variation of g (i.e. a vector field along g). The vector field on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ generated be the element $(\partial, H) \in \mathrm{Lie}_{\mathbb{C}}(S^1 \times S^1 \times T)$ is given at the point $gT_{\mathbb{C}}$ by $\partial g + \mathrm{ad}\, H(g)$. So we have to show that

$$-\mathrm{d}S_{G,H,1}(\delta g) = \omega^H_{gT}(\delta g, \partial g + \mathrm{ad}\, H(g)).$$

Let us denote the H-dependent term in $S_{G,H,1}(g)$ by $\tilde{S}_H(g)$ so that we have

$$S_{G,H,1}(g) = S_{G,1} + \tilde{S}_H(g).$$

From the Polyakov–Wiegmann formula (Proposition 4.1) one deduces that

$$-\mathrm{d}S_{G,1}(\delta g) = \frac{\pi}{\tau_2}\int_{\Sigma} \langle g^{-1}\partial g, \bar{\partial}(g^{-1}\delta g)\rangle\, \mathrm{d}s\, \mathrm{d}t.$$

On the other hand, we have

$$-\mathrm{d}\tilde{S}_H(\delta g) = \frac{\pi}{\tau_2}\int_{\Sigma} (\langle \delta(g^{-1}\partial g), H\rangle - \langle \delta(\bar{\partial} g g^{-1}), H\rangle - \langle H, \delta(g^{-1}Hg\rangle)\, \mathrm{d}s\, \mathrm{d}t.$$

We already know that $\delta(g^{-1}\partial g) = \partial(g^{-1}\delta g) + [g^{-1}\partial g, g^{-1}\delta g]$. Thus, by partial integration, $\int_{\Sigma}\langle \delta(g^{-1}\partial g), H\rangle = \int_{\Sigma}\langle [g^{-1}\partial g, g^{-1}\delta g], H\rangle\, \mathrm{d}s\, \mathrm{d}t$. Short calculations show $\langle \delta(\bar{\partial} g g^{-1}), H\rangle = \langle \bar{\partial}(g^{-1}\delta g), g^{-1}Hg\rangle$ and $\langle H, \delta(g^{-1}Hg)\rangle = \langle H, [g^{-1}Hg, g^{-1}\delta g]\rangle$. Putting all these terms together gives

$$\begin{aligned} -\mathrm{d}S_{G,H,1}(\delta g) = {} & \frac{\pi}{\tau_2}\int_{\Sigma} \langle \bar{\partial}(g^{-1}\delta g), g^{-1}\partial g + g^{-1}Hg\rangle\, \mathrm{d}s\, \mathrm{d}t \\ & + \frac{\pi}{\tau_2}\int_{\Sigma} \langle H, [g^{-1}\delta g, g^{-1}\partial g - g^{-1}Hg]\rangle\, \mathrm{d}s\, \mathrm{d}t, \end{aligned}$$

which is the assertion. □

Since we have $S_{G,H,\kappa} = \kappa S_{G,H,1}$, Lemma 4.4 allows us to calculate the integral $\int_{LLG_{\mathbb{C}}/T_{\mathbb{C}}} \mathrm{e}^{S_{G,H,\kappa}}\, \mathrm{d}\sigma$ via the Liouville functional approach. Before we start with the calculation, let us briefly recall some facts from the theory of affine Lie algebras (see [12] for details). Let $\tilde{\mathfrak{g}}_{\mathbb{C}} = L\mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the untwisted affine Lie algebra corresponding to $\mathfrak{g}_{\mathbb{C}}$ and let A denote its generalized Cartan matrix. Let $\tilde{\Delta}$ denote the root system of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ and choose a set $\alpha_0 \ldots, \alpha_l$ of simple roots. Denote by $\alpha_0^{\vee} \ldots, \alpha_l^{\vee}$ the dual simple roots, i.e. $\alpha_i \in \tilde{\mathfrak{h}}$ such that $\langle \alpha_i, \alpha_j^{\vee}\rangle = (A)_{i,j}$. Let a_i be the "minimal" integers such that $A(a_0, \ldots, a_n) = 0$ and set $\delta = \sum_{i=0}^{n} a_i\alpha_i$. Following [12, Section 12.4], we define the canonical central element of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ as

$$K = \sum_{i=0}^{l} a_i \alpha_i^{\vee}.$$

Turning to the representation theory of affine Lie algebras we define as usual

$$\begin{aligned}\tilde{P} &= \{\lambda \in \tilde{\mathfrak{h}}^*_{\mathbb{C}} | \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } i = 0, \ldots, n\},\\ \tilde{P}_+ &= \{\lambda \in \tilde{P} | \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i = 0, \ldots, n\}, \text{ and}\\ \tilde{P}_{++} &= \{\lambda \in \tilde{P} | \langle \lambda, \alpha_i^\vee \rangle > 0 \text{ for all } i = 0, \ldots, n\}.\end{aligned}$$

There exists a bijection between the irreducible integrable highest weight modules of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ and the dominant integral weights $\lambda \in \tilde{P}_+$. For $\lambda \in P_+$ let $L(\lambda)$ denote the corresponding irreducible integrable highest weight module of $\tilde{\mathfrak{g}}_{\mathbb{C}}$. There is no essential loss in generality for the representation theory of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ if we assume $\langle \lambda, d \rangle = 0$ for a highest weight λ of $\tilde{\mathfrak{g}}_{\mathbb{C}}$, so from now on, we restrict $\tilde{P}$ to $\{\lambda \text{ s.t.} \langle \lambda, d \rangle = 0\}$.

Since the highest weight modules $L(\lambda)$ are irreducible, K operates as a scalar on $L(\lambda)$. We define the level k of λ to be the non-negative integer $k = \langle \lambda, K \rangle$ and set

$$\tilde{P}^k_+ = \{\lambda \in \tilde{P}_+ \text{ s.t. level}(\lambda) = k\}.$$

Now for $\lambda \in \tilde{P}_+$ let $\mathrm{ch}(\lambda)$ be the character and χ_λ be the normalized character of the $\tilde{\mathfrak{g}}_{\mathbb{C}}$-module $L(\lambda)$ (see [12, Chapter 10 and Section 12.7]). Choose an element $\Lambda_0 \in \tilde{\mathfrak{h}}$ such that $\langle \Lambda_0, \alpha_0 \rangle = 1$ and $\langle \Lambda_0, \Lambda_0 \rangle = \langle \Lambda_0, \alpha_i^\vee \rangle = 0$ for all $i = 1, \ldots, l$. If we choose orthonormal coordinates $v_1, \ldots, v_l$ of $\mathfrak{h}$ (with respect to the negative Killing form on $\mathfrak{g}$), we can coordinatize $\mathfrak{h}_{\mathbb{C}}$ via

$$v = 2\pi \mathrm{i} \left(\sum_{\nu=1}^{l} z_\nu v_\nu - \tau \Lambda_0 + u\delta \right),$$

and identify $v \in \tilde{\mathfrak{h}}_{\mathbb{C}}$ with the vector (τ, H, u) with $H = \sum z_\nu v_\nu \in \mathfrak{h}_{\mathbb{C}}$ and $\tau, u \in \mathbb{C}$.

It is known (see e.g. [12]), that for any $\lambda \in \tilde{P}_+$, the character $\mathrm{ch}(\lambda)$ and the normalized character χ_λ converge absolutely on the domain

$$Y = \{(\tau, H, u) | H \in \mathfrak{h}_{\mathbb{C}}; \tau, u \in \mathbb{C}, \mathrm{Im}(\tau) > 0\}.$$

Hence $\mathrm{ch}(\lambda)$ and χ_λ define holomorphic functions on Y. Since the center of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ acts on $L(\lambda)$ by scalar multiplication, we can view $\mathrm{ch}(\lambda)$ and χ_λ as functions $\mathrm{ch}(\lambda)(\tau, H)$ and $\chi_\lambda(\tau, H)$ of τ and H and forget about the central u-coordinate without loss of information. For a geometric interpretation of the $\chi_\lambda(\tau, H)$ as sections of certain line bundles over certain Abelian varieties, see e.g. [6,17].

An explicit formula for the normalized character $\chi_\lambda(\tau, H)$ is given by the Kac–Weyl character formula: As before, let $Q^\vee \subset \mathfrak{h}$ be the dual root lattice of $\mathfrak{g}_{\mathbb{C}}$ (with the appropriate identifications) and let Δ_+ and $\tilde{\Delta}_+$ be the set of positive roots of $\mathfrak{g}_{\mathbb{C}}$ and $\tilde{\mathfrak{g}}_{\mathbb{C}}$, respectively (with respect to the simple roots, $\alpha_0, \ldots, \alpha_l$). Set $\rho = \frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha$. For $\mu \in \tilde{P}$ define $\bar{\mu} \in \mathfrak{h}$ to be the projection of μ to $\mathfrak{h}$ and for $x, \tau \in \mathbb{C}$, set $e(x) = \mathrm{e}^{2\pi \mathrm{i} x}$ and $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$. Then for $\lambda \in \tilde{P}^k_+$ C define

$$\Theta_\lambda(\tau, H) = \sum_{\gamma \in Q^\vee + k^{-1}\bar{\lambda}} e\left(\frac{1}{2} k\tau \langle \gamma, \gamma \rangle + k \langle \gamma, H \rangle \right).$$

With these definitions, the Kac–Weyl character formula (cf. [12, Chapters 10 and 12]) reads

$$\chi_\lambda(\tau, H) = \frac{\sum_{w\in W}(-1)^{l(w)}\Theta_{w(\lambda+\tilde\rho)}(\tau, H)}{q^{\dim\mathfrak{g}/24}e(\langle\rho, H\rangle)\prod_{\alpha\in\tilde\Delta_+}(1-e(-\alpha(\tau, H)))^{\mathrm{mult}\,\alpha}},$$

where $\mathrm{mult}\,\alpha = \dim\mathfrak{g}_\alpha$ denotes the dimension of the root space corresponding to $\alpha\in\tilde\Delta$, and $\tilde\rho\in\mathfrak{h}^*$ is defined via $\langle\tilde\rho, \alpha_i^\vee\rangle = 1$ for $i = 0, \dots, l$ and $\langle\tilde\rho, d\rangle = 0$.

The squared absolute value of the denominator of this formula is, up to the coefficient C, exactly the product $\mathrm{Pf}_\zeta(B_\sigma)Z(H)$ calculated in Proposition 4.2. This can be seen using the decomposition of $\tilde\Delta_+$ into real and imaginary roots

$$\tilde\Delta_+ = \Delta_+ \cup \bigcup_{n=1}^{\infty}\{\Delta + n\delta\} \cup \bigcup_{n=1}^{\infty} n\delta.$$

$\Delta_+^{\mathrm{im}} = \{n\delta | n\in\mathbb{N}\}$ is called the set of positive imaginary roots. The multiplicities of the roots are given by $\mathrm{mult}\,\alpha = 1$ for $\alpha\in\Delta_+ - \Delta_+^{\mathrm{im}}$ and $\mathrm{mult}\,\alpha = l$ for Δ_+^{im}.

We can now state the main theorem of this section.

Theorem 4.5. *Let $h^\vee$ denote the dual Coxeter number of $\mathfrak{g}_\mathbb{C}$ and let k be a positive level of $\tilde{\mathfrak{g}}_\mathbb{C}$. Set $\kappa = h^\vee + k$. Then the following identity is valid*:

$$\int_{LLG_\mathbb{C}/T_\mathbb{C}} \mathrm{e}^{S_{G,H,\kappa}}\,\mathrm{d}\sigma = \frac{C_0}{C_1 C}\sum_{\lambda\in\tilde P_+^k}|\chi_\lambda(\tau, H)|^2$$

with $C_0 = \kappa^l$, $C_1 = \sqrt{2\kappa\tau_2}^{\,1}/\mathrm{vol}\,\kappa Q^\vee$ and C as in Proposition 4.2.

Proof. The equality of the denominators in the equation follows with Proposition 4.2 and the remarks after the Kac–Weyl character formula. So we have to calculate $S_{G,H,\kappa}$ in the fixed points of the $S^1\times S^1\times T$-action which are given by $(s,t)\mapsto g_w\exp(s\beta)\exp(t\mu)$ with $\beta, \mu\in Q^\vee$ and where g_w is a representative of $w\in W$. The action function $S_{G,H,\kappa}$ can be rewritten as

$$S_{G,H,\kappa}(g) = -\frac{\kappa\pi}{2\tau_2}\int_\Sigma (\langle g^{-1}\partial g + g^{-1}Hg - H, g^{-1}\bar\partial g + g^{-1}Hg - H\rangle$$
$$- 2\mathrm{i}\langle g^{-1}\partial_t g, g^{-1}Hg + H\rangle)\,\mathrm{d}s\,\mathrm{d}t + \frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}(\tilde g^{-1}\,\mathrm{d}\tilde g)^{\wedge 3}.$$

For $g(s,t) = g_w\exp(s\beta)\exp(t\mu)$, the Wess–Zumino term is easily calculated using its translation quasi-invariance. In the proof of Lemma 4.1 we saw that

$$\frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}((gh)^{-1}\mathrm{d}(gh))^{\wedge 3} = \frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}(g^{-1}\mathrm{d}g)^{\wedge 3} + \frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}(h^{-1}\,\mathrm{d}h)^{\wedge 3}$$
$$+\mathrm{i}\kappa\pi\int_\Sigma \mathrm{tr}(g^{-1}\,\mathrm{d}g\wedge\mathrm{d}hh^{-1}).$$

Writing $g(s,t) = g_w(\exp(s\beta)\exp(t\mu))$, we see that the constant term g_w does not contribute to the integral. Furthermore, we have

$$\frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}((\exp(s\beta)\exp(t\mu))^{-1}\,\mathrm{d}(\exp(s\beta)\exp(t\mu)))^{\wedge 3}$$
$$= \frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}(\exp(-s\beta)\,\mathrm{d}(\exp(s\beta)))^{\wedge 3} + \frac{\mathrm{i}\kappa\pi}{3}\int_B \mathrm{tr}(\exp(-t\mu)\,\mathrm{d}(\exp(t\mu)))^{\wedge 3}$$
$$+\mathrm{i}\kappa\pi\int_\Sigma \mathrm{tr}(\exp(-s\beta)\,\mathrm{d}(\exp(s\beta))\wedge \mathrm{d}(\exp(t\mu))\exp(-t\mu)).$$

The first two terms of the left-hand side of the equation vanish and the third term is easily calculated to be $\kappa\pi\mathrm{i}\langle\beta,\mu\rangle$. So in the fixed points of the torus-action, the action function reads

$$S_{G,H,\kappa}(g_w\exp(s\beta)\exp(t\mu)) = -\frac{\pi\kappa}{2\tau_2}\langle\beta+\tau\mu+w^{-1}H-H,\beta+\bar\tau\mu+w^{-1}H-H\rangle$$
$$+\pi\mathrm{i}\kappa\langle\mu,w^{-1}H+H\rangle - \pi\mathrm{i}\kappa\langle\beta,\mu\rangle.$$

Now after rearranging the order of the summation, Theorem 4.5 follows with Lemma 4.6. □

Lemma 4.6. *The following identity is valid:*

$$\frac{\mathrm{vol}\,\kappa Q^\vee}{\sqrt{2\kappa\tau_2}^{\,l}}\sum_{\beta,\mu\in Q^\vee}\sum_{w\in W}(-1)^{l(w)}\,\mathrm{e}^{-(\pi\kappa/2\tau_2)\langle\beta+\tau\mu+H-wH,\beta+\bar\tau\mu+H-wH\rangle}$$
$$\times\mathrm{e}^{\pi\mathrm{i}\kappa\langle\beta,\mu\rangle+\pi\mathrm{i}\kappa\langle\mu,H+wH\rangle}$$
$$=\sum_{\lambda\in\tilde P_+^\kappa}\sum_{w_1\in W}(-1)^{l(w_1)}\Theta_{w_1(\lambda+\tilde\rho)}(\tau,H)\sum_{w_2\in W}(-1)^{l(w_2)}\overline{\Theta_{w_2(\lambda+\tilde\rho)}(\tau,H)}.$$

Proof. Let us denote the right-hand side of the equation in Lemma 4.6 with $N_k(\tau,H)$. Note that we have $\bar{\tilde\rho}=\rho$ and level $(\tilde\rho)=h^\vee$ (see [12, Chapter 12]). Therefore

$$N_\kappa(\tau,H)=\sum_{\lambda\in\tilde P_+^\kappa}\sum_{w_1\in W}(-1)^{l(w_1)}\Theta_{w_1(\lambda+\tilde\rho)}(\tau,H)\sum_{w_2\in W}(-1)^{l(w_2)}\overline{\Theta_{w_2(\lambda+\tilde\rho)}(\tau,H)}$$
$$=\sum_{\lambda\in\tilde P_{++}^\kappa}\sum_{w_1\in W}\sum_{\gamma\in Q^\vee+(1/\kappa)w_1\bar\lambda}(-1)^{l(w_1)}e\left(\frac{1}{2}\kappa\tau\langle\gamma,\gamma\rangle+\kappa\langle\gamma,H\rangle\right)$$
$$\times\sum_{w_2\in W}(-1)^{l(w_2)}\overline{\Theta_{w_2(\lambda+\tilde\rho)}(\tau,H)}$$
$$=\sum_{\lambda\in\tilde P_{++}^\kappa}\sum_{\alpha\in Q^\vee}\sum_{w,w'\in W}(-1)^{l(w')}(-1)^{l(ww')}e\left(\frac{1}{2}\kappa\tau\left\langle\frac{1}{\kappa}w'\bar\lambda+\alpha,\frac{1}{\kappa}w'\bar\lambda+\alpha\right\rangle\right)$$
$$\times e\left(k\left\langle\frac{1}{\kappa}w'\bar\lambda+\alpha,H\right\rangle\right)\overline{\Theta_{w'\bar\lambda+\alpha}(\tau,wH)}.$$

The set $\{(1/\kappa)\bar{\lambda}|\lambda \in \tilde{P}^{\kappa}_{++}\}$ lies in a fundamental alcove of the affine Weyl group and the singular weights do not contribute to the sum below. Since Θ_γ only depends on the class of γ modulo $Q^\vee$, we can sum over $\alpha \in Q^\vee$ and $w' \in W$ and get

$$= \sum_{\gamma\in(1/\kappa)P}\sum_{w\in W}(-1)^{l(w)}e\left(\frac{1}{2}\tau\kappa\langle\gamma,\gamma\rangle+\kappa\langle\gamma,H\rangle\right)\overline{\Theta_\gamma(\tau,wH)}$$

$$= \sum_{\gamma\in(1/\kappa)P}\sum_{w\in W}\sum_{\mu\in Q^\vee}(-1)^{l(w)}e\left(\frac{1}{2}\tau\kappa\langle\gamma,\gamma\rangle+\kappa\langle\gamma,H\rangle\right)$$

$$\times e\left(-\frac{1}{2}\bar{\tau}\kappa\langle\gamma+\mu,\gamma+\mu\rangle-\kappa\langle\gamma+\mu,wH\rangle\right)$$

$$= \sum_{\gamma\in(1/\kappa)P}\sum_{w\in W}\sum_{\mu\in Q^\vee}(-1)^{l(w)}\,\mathrm{e}^{-2\pi\kappa\tau_2\langle\gamma,\gamma\rangle-2\pi\mathrm{i}\kappa\bar{\tau}\langle\gamma,\mu\rangle-\pi\mathrm{i}\kappa\bar{\tau}\langle\mu,\mu\rangle}$$

$$\times\mathrm{e}^{2\pi\mathrm{i}\kappa\langle\gamma,H\rangle-2\pi\mathrm{i}\kappa\langle\gamma+\mu,wH\rangle}$$

$$= \sum_{\gamma\in(1/\kappa)P}\sum_{w\in W}\sum_{\mu\in Q^\vee}(-1)^{l(w)}\,\mathrm{e}^{-2\pi\kappa\tau_2(\langle\gamma,\gamma\rangle+\langle\gamma,\mu\rangle)-2\pi\mathrm{i}\kappa\tau_1\langle\gamma,\mu\rangle+2\pi i\kappa\langle\gamma,H-wH\rangle}$$

$$\times\mathrm{e}^{-\pi\mathrm{i}\bar{\tau}\kappa\langle\mu,\mu\rangle-2\pi\mathrm{i}\kappa\langle\mu,wH\rangle}.$$

We now need to apply the Poisson transformation formula: For an Euclidean vector space V, a lattice $M\subset V$ and a Schwartz function $f: V\to\mathbb{C}$ one has

$$\sum_{\beta\in M^\vee}\hat{f}(\beta)=\mathrm{vol}\,M\sum_{\gamma\in M}f(\gamma)$$

with

$$\hat{f}(\beta)=\int_V\mathrm{e}^{2\pi\mathrm{i}\langle\gamma,\beta\rangle}f(\gamma)\,\mathrm{d}\gamma,$$

and where $M^\vee$ denotes the dual lattice of M with respect to the scalar product on V. If we choose

$$f(\gamma)=\mathrm{e}^{-2\pi\kappa\tau_2(\langle\gamma,\gamma\rangle+\langle\gamma,\mu\rangle)-2\pi\mathrm{i}\kappa\tau_1\langle\gamma,\mu\rangle+2\pi\mathrm{i}\kappa\langle\gamma,H-wH\rangle-\pi\mathrm{i}\bar{\tau}\kappa\langle\mu,\mu\rangle-2\pi\mathrm{i}\kappa\langle\mu,wH\rangle},$$

a direct calculation yields

$$\hat{f}(\beta)=\frac{1}{\sqrt{2\kappa\tau_2}}$$
$$\times\mathrm{e}^{-\pi\mathrm{i}\langle\mu,\beta\rangle-(\pi/2\tau_2\kappa)\langle\beta,\beta\rangle+(\pi\tau_1/\tau_2)\langle\beta,\mu\rangle-(\pi/\tau_2)\langle\beta,H-wH\rangle-(\pi(\tau_1^2+\tau_2^2)\kappa/2\tau_2)\langle\mu,\mu\rangle}$$
$$\times\mathrm{e}^{(\pi\tau_1\kappa/\tau_2)\langle\mu,H-wH\rangle-(\pi\kappa/2\tau_2)\langle H-wH,H-wH\rangle-\pi\mathrm{i}\kappa\langle\mu,H+wH\rangle}.$$

So by the Poisson summation formula, we get

$$
\begin{aligned}
N_\kappa(\tau, H) &= \frac{1}{\mathrm{vol}(1/\kappa) P \sqrt{2\kappa\tau_2}^{\,l}} \sum_{\beta\in\kappa Q^\vee} \sum_{w\in W} \sum_{\mu\in Q^\vee} (-1)^{l(w)}\, \mathrm{e}^{-\pi\mathrm{i}\langle\mu,\beta\rangle-(\pi/2\tau_2\kappa)\langle\beta,\beta\rangle} \\
&\quad \times \mathrm{e}^{(\pi\tau_1/\tau_2)\langle\beta,\mu\rangle-(\pi/\tau_2)\langle\beta,H-wH\rangle-(\pi(\tau_1^2+\tau_2^2)\kappa/2\tau_2)\langle\mu,\mu\rangle+(\pi\tau_1\kappa/\tau_2)\langle\mu,H-wH\rangle} \\
&\quad \times \mathrm{e}^{-(\pi\kappa/2\tau_2)\langle H-wH,H-wH\rangle-\pi\mathrm{i}\kappa\langle\mu,H+wH\rangle} \\
&= \frac{\mathrm{vol}\,\kappa Q^\vee}{\sqrt{2\kappa\tau_2}^{\,l}} \sum_{\beta,\mu\in Q^\vee} \sum_{w\in W} (-1)^{l(w)}\, \mathrm{e}^{-\pi\mathrm{i}\kappa\langle\mu,\beta\rangle-(\pi\kappa/2\tau_2)\langle\beta,\beta\rangle} \\
&\quad \times \mathrm{e}^{(\pi\tau_1\kappa/\tau_2)\langle\beta,\mu\rangle-(\pi\kappa/\tau_2)\langle\beta,H-wH\rangle-(\pi(\tau_1^2+\tau_2^2)\kappa/2\tau_2)\langle\mu,\mu\rangle+(\pi\tau_1\kappa/\tau_2)\langle\mu,H-wH\rangle} \\
&\quad \times \mathrm{e}^{-(\pi\kappa/2\tau_2)\langle H-wH,H-wH\rangle-\pi\mathrm{i}\kappa\langle\mu,H+wH\rangle} \\
&= \frac{\mathrm{vol}\,\kappa Q^\vee}{\sqrt{2\kappa\tau_2}^{\,l}} \sum_{\beta,\mu\in Q^\vee} \sum_{w\in W} (-1)^{l(w)}\, \mathrm{e}^{-(\pi\kappa/2\tau_2)\langle\beta-\tau\mu+H-wH,\beta-\bar\tau\mu+H-wH\rangle} \\
&\quad \times \mathrm{e}^{-\pi\mathrm{i}\kappa\langle\beta,\mu\rangle-\pi\mathrm{i}\kappa\langle\mu,H+wH\rangle} \\
&= \frac{\mathrm{vol}\,\kappa Q^\vee}{\sqrt{2\kappa\tau_2}^{\,l}} \sum_{\beta,\mu\in Q^\vee} \sum_{w\in W} (-1)^{l(w)}\, \mathrm{e}^{-(\pi\kappa/2\tau_2)\langle\beta+\tau\mu+H-wH,\beta+\bar\tau\mu+H-wH\rangle} \\
&\quad \times \mathrm{e}^{\pi\mathrm{i}\kappa\langle\beta,\mu\rangle+\pi\mathrm{i}\kappa\langle\mu,H+wH\rangle}.
\end{aligned}
$$

This finishes the proof of Lemma 4.6. □

The modular group $SL(2,\mathbb{Z})$ acts naturally on the torus $S^1 \times S^1$. Under this action, the modular parameter τ of the elliptic curve Σ_τ is transformed via

$$
\tau \mapsto \frac{a\tau+b}{c\tau+d}
$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. This $SL(2,\mathbb{Z})$-action can be extended to the domain Y via

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\tau, H, u) \mapsto \left(\frac{a\tau+b}{c\tau+d}, \frac{H}{c\tau+d}, u - \frac{c\langle H, H\rangle}{2(c\tau+d)} \right).
$$

It was shown in [13] that for each $k \in \mathbb{N}$, the $SL(2,\mathbb{Z})$-action on Y constructed above gives rise to an $SL(2,\mathbb{Z})$-action on the set of normalized characters of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ at level k. In particular, it follows from the explicit transformation properties of the characters under the $SL(2,\mathbb{Z})$-action that the sum $\sum_{\lambda\in\tilde{P}_+^\kappa} |\chi_\lambda(\tau, H)|^2$ is $SL(2,\mathbb{Z})$-invariant. Since the $SL(2,\mathbb{Z})$-action on the modular parameter arises naturally in the functional integral setup considered in this paper, one should expect that the $SL(2,\mathbb{Z})$-invariance of the partition function at level k can easily be deduced from its representation as a functional integral. In fact, the search for a derivation of the $SL(2,\mathbb{Z})$-invariance of the partition function using only functional integrals was one of the starting points of this paper. Unfortunately, this does not seem possible since for the zeta regularization to work, $\alpha(H)$ has to be a real parameter for all $\alpha \in \Delta$. But the $SL(2,\mathbb{Z})$-action on Y does not leave the space $\{H | \alpha(H) \in \mathbb{R}$ for all $\alpha \in \Delta\}$ invariant. We can only deduce a slightly weaker result.

The sum $\sum_{\lambda\in\tilde{P}_+^\kappa}|\chi_\lambda(\tau,H)|^2$ depends continuously on $H\in\mathfrak{h}$ and is well defined for $H=0$. The functional integral in Theorem 4.5 is defined for all generic $H\in\mathfrak{h}$ and from the equality proved in the theorem we can deduce that it is a continuous function of H as well. Thus, we can formally write

$$\int_{LLG_{\mathbb{C}}/T_{\mathbb{C}}} \mathrm{e}^{S_{G,0,\kappa}}\,\mathrm{d}\sigma = \frac{C_0}{C_1 C_{\tau,H}}\sum_{\lambda\in\tilde{P}_+^\kappa}|\chi_\lambda(\tau,0)|^2.$$

Now one can easily check that the denominator and the numerator of the left-hand side in the equation above are $SL(2,\mathbb{Z})$-invariant, and since the τ_2-dependent terms in the coefficients of the right-hand side cancel, we have the following corollary of Theorem 4.5.

Corollary 4.7.

$$\sum_{\lambda\in\tilde{P}_+^\kappa}|\chi_\lambda(\tau,0)|^2$$

is a modular invariant function of τ.

Let us mention a slight generalization of the WZW model: Analogously to Section 3.5, one can define a twisted version of the WZW model by twisting the loop directions with an outer automorphism ψ of order $\mathrm{ord}(\psi)=r$ of the group $G_{\mathbb{C}}$. That is, we consider the space

$$LL(G_{\mathbb{C}},\psi) = \left\{g\in LLG_{\mathbb{C}}|\psi(g(s,t))=g\left(s,t+\frac{1}{r}\right)=g\left(s+\frac{1}{r},t\right)\ \text{ for all }\ t\in[0,1]\right\}.$$

For $H\in T^\psi$, we can restrict the action function $S_{G,H,\kappa}$ and the corresponding $\mathbb{R}$-action on $LG_{\mathbb{C}}/T_{\mathbb{C}}$ to obtain an action functional and an $\mathbb{R}$-action on $LL(G_{\mathbb{C}},\psi)/T_{\mathbb{C}}^\psi$. Now we can use the Liouville functional to calculate the formal integral

$$\int_{LL(G_{\mathbb{C}},\psi)/T_{\mathbb{C}}^\psi} \mathrm{e}^{S_{G,H,\kappa}(g)\,\mathrm{d}\sigma(g)}.$$

For simplicity, let us exclude the case that $G_{\mathbb{C}}$ is of type A_{2l}. Then similar calculations as in Sections 3.5 and 4.3 show that the partition function of the twisted WZW model is given by

$$\int_{LL(G_{\mathbb{C}},\psi)/T_{\mathbb{C}}^\psi} \mathrm{e}^{S_{G,H,\kappa}(g)}\mathrm{d}\sigma(g) = C_\psi\sum_{\lambda\in\tilde{P}_+^{\psi\vee,\kappa}}|\chi_\lambda(\tau,H)|^2$$

with $\kappa=k+h^\vee$ as before. Here $C_\psi\in\mathbb{R}$ is some constant, and $P_+^{\psi\vee,\kappa}$ denotes the set of highest weights of level k of the twisted affine Lie algebra $\tilde{g}_{\mathbb{C}}^{\psi,\vee}$, whose finite-dimensional root system is dual to the root system of the finite-dimensional Lie algebra $\mathfrak{g}_{\mathbb{C}}^\psi$. For example, if $g_{\mathbb{C}}$ is of type D_l, then $\tilde{g}_{\mathbb{C}}^{\psi,\vee}$ is the twisted affine Lie algebra of type $\mathrm{A}_{2(l-1)-1}^{(2)}$ and accordingly in the other cases. This is the same "duality" between root systems which appears in the calculation of the irreducible characters of certain non-connected compact Lie groups [27].

In any case, the largest subgroup of $SL(2,\mathbb{Z})$ acting on $LL(G_{\mathbb{C}},\psi)$ is the congruence subgroup

$$\Gamma(r)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})|a\equiv b\equiv 1 \bmod r \text{ and } b\equiv c\equiv 0 \bmod r\right\}.$$

Now a similar argument as above yields the $\Gamma(r)$-invariance of the partition function of the twisted WZW model at $H=0$.

4.4. Concluding remarks

Let us close this exposition with some remarks about a measure theoretic interpretation of the calculations leading to the partition function of the WZW model at level κ. As we saw in Section 3.4, the partition function of the compact Lie group G can be expressed as an integral over the space of continuous paths $\gamma:[0,1]\to G$ such that $\gamma(0)=e$ and $\gamma(1)\in\mathcal{O}_{\exp(H)}$. One of the main ingredients in this interpretation is the observation that the coadjoint orbits of the centrally extended loop group $\hat{G}$ can be classified in terms of conjugacy classes of the group G. This observation generalizes directly to the case of double loop groups.

Let $L_{\mathrm{h}}G_{\mathbb{C}}$ be the holomorphic loop group, i.e. the set of holomorphic maps from the cylinder $\mathbb{C}/\mathbb{Z}$ to $G_{\mathbb{C}}$. The group $\mathbb{C}/\mathbb{Z}$ acts on $L_{\mathrm{h}}G_{\mathbb{C}}$ by automorphisms, and we denote the semi-direct product $\mathbb{C}/\mathbb{Z}\ltimes L_{\mathrm{h}}G_{\mathbb{C}}$ by $\check{G}_{\mathbb{C}}$. Let $(\tau,\delta)\in\check{G}_{\mathbb{C}}$. Then a short calculation shows that τ is invariant under conjugation with any element $(z,\gamma_1)\in\check{G}_{\mathbb{C}}$. Thus, the coset $C_\tau=\{(\tau,\gamma)|\gamma\in L_{\mathrm{h}}G_{\mathbb{C}}\}$ is fixed under conjugation. The group $LLG_{\mathbb{C}}$ admits a one-dimensional central extension $LL\hat{G}_{\mathbb{C}}$. Now the corresponding coadjoint orbits of $LL\hat{G}_{\mathbb{C}}$ (resp. the affine coadjoint orbits of $LLG_{\mathbb{C}}$) can be classified in terms of $L_{\mathrm{h}}G_{\mathbb{C}}$-conjugacy classes inside C_τ in the same way the coadjoint orbits of $\hat{G}$ are classified in terms of conjugacy classes of G (see [5]).

Thus, in order to generalize the Wiener measure approach to the calculation of the partition function on a compact Lie group in Section 3.4 to the WZW model, one has to develop the notion of Brownian motion on the holomorphic loop group $L_{\mathrm{h}}G_{\mathbb{C}}$ or an appropriate completion thereof. It does not seem unlikely that such a generalization is possible since important ingredients for the construction of the Wiener measure on the compact group G like the Haar measure and the Laplacian admit generalizations to the case of Kac–Moody groups (see e.g. [6,20]). The generalization of the Wiener measure approach to functional integrals would be very interesting since it would provide solid mathematical ground for dealing with such integrals.

Of course, even such a measure theoretic reinterpretation of our calculation of the WZW partition function would still leave open a much more fundamental question: The Duistermaat Heckman formula is a well-established result for finite-dimensional symplectic manifolds. As realized in the physics literature [1,19] and in this paper, a straightforward generalization of the formalism to certain infinite-dimensional manifolds gives interesting results which in some cases can also be derived by usual integration methods. So it would be interesting to know how a class of infinite-dimensional manifolds could look like on which a stringent measure theory can be developed which includes a version of the Duistermaat Heckman formula as a theorem. As we saw in Section 3.4, an appropriate closure of the coadjoint orbits of the loop group LG should certainly belong to such a class.

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Appendix A. Zeta-regularized products

In this section, we will recall the definition and some basic properties of zeta-regularized products. See e.g. [11] for a more comprehensive introduction to the theory of regularized products. To motivate the following definition, recall that the product of $\lambda_1, \dots, \lambda_N \in \mathbb{R}_{>0}$, can be written as

$$\prod_{n=1}^{N}\lambda_n = \exp\left(-\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\sum_{n=0}^{N}\frac{1}{\lambda_n^s}\right).$$

Definition A.1. Let $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ be a sequence of positive real numbers. For $s \in \mathbb{C}$, define

$$\zeta_\Lambda(s) = \sum_{n=1}^{\infty}\frac{1}{\lambda_n^s}.$$

If $\zeta_\Lambda(s)$ converges for $\mathrm{Re}(s)$ sufficiently large and if the function ζ_Λ can be analytically continued to a meromorphic function on $\mathbb{C}$ which is regular at $s = 0$, we define the zeta-regularized product of $\{\lambda_1, \lambda_2, \dots\}$ by

$$\left(\prod_{n=1}^{\infty}\lambda_n\right)_\zeta = \exp(-\zeta_\Lambda'(0)),$$

and call such a sequence zeta-multipliable.

Note that if the usual limit $\prod_{n=1}^{\infty}\lambda_n$ exists, then $\lim_{n\to\infty}\lambda_n = 1$. Thus the corresponding zeta function does not converge anywhere. Two important properties of the zeta-regularized product are stated in the following lemma.

Lemma A.2. *Let $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ be a zeta-multipliable sequence and $a, b \in \mathbb{R}_{>0}$. Then the sequences $\{\lambda_1^a, \lambda_2^a, \dots\}$ and $\{\lambda_1^b, \lambda_2^b, \dots\}$ are zeta-multipliable and we have*

$$\left(\prod_{n=1}^{\infty}\lambda_n^a\right)_\zeta = \left[\left(\prod_{n=1}^{\infty}\lambda_n\right)_\zeta\right]^a,$$

and

$$\left(\prod_{n=1}^{\infty}\lambda_n^b\right)_\zeta = \left(\prod_{n=1}^{\infty}\lambda_n\right)_\zeta b^{\zeta_\Lambda(0)}.$$

Let us state the examples of zeta-regularized products which were used in Chapters 3 and 4: First consider the Riemann zeta function which is given by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Obviously, $\zeta(s)$ converges for $\mathrm{Re}(s) > 1$. Furthermore, ζ can be analytically continued to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ and $\zeta(0) = \frac{1}{2}$. Standard theory of the Riemann zeta function (cf. [15,16]) implies $\zeta'(0) = -\log\sqrt{2\pi}$. Thus, the sequence $\{1, 2, 3, \dots\}$ is zeta-multipliable and we have

$$\left(\prod_{n=1}^{\infty} n\right)_\zeta = \sqrt{2\pi}.$$

As a second example, we consider a class of zeta functions, the so-called Epstein zeta functions.

Definition A.3. Let $\tau_1, \tau_2, v \in \mathbb{R}$ and $\tau = \tau_1 + \mathrm{i}\tau_2 \in \mathbb{C}$ such that $\tau_2 > 0$. The Epstein zeta functions are given by

$$\zeta_\tau(s; v) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)+(v,0)\neq 0}} \frac{\tau_2^s}{|m + \tau n + v|^{2s}}$$

and

$$\tilde{\zeta}_\tau(s; v) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{\tau_2^s\,\mathrm{e}^{2\pi\mathrm{i}mv}}{|m + \tau n|^{2s}}.$$

The series defining ζ_τ and $\tilde{\zeta}_\tau$ converge absolutely for $s > 1$ and define analytic functions on $\{s \in \mathbb{R}, s > 1\}$. The most important properties of ζ_τ and $\tilde{\zeta}_\tau$ are stated in the following theorem.

Theorem A.4. *The functions $\zeta_\tau(s; v)$ and $\tilde{\zeta}_\tau(s; v)$ have analytic continuations to the entire s-plane. If $v \notin \mathbb{Z}$, the continuations are entire functions of s. If $v \in \mathbb{Z}$, then $\zeta_y(s; v)$ and*

$\tilde{\zeta}_\tau(s; v)$ are meromorphic in the entire s-plane with the only singularity at $s = 1$. In all cases, ζ_τ and $\tilde{\zeta}_\tau$ satisfy the functional equation

$$\pi^{-s}\Gamma(s)\zeta_\tau(s; v) = \pi^{-(1-s)}\Gamma(1-s)\tilde{\zeta}_\tau(1-s; v),$$

where $\Gamma(s)$ is the usual Γ-function.

For a proof of Theorem A.4 as well as an exposition of the theory of much more general Epstein zeta functions, see [24].

Using the functional equation satisfied by ζ_τ and $\tilde{\zeta}_\tau$, one can prove the Kronecker limit formulas (see [16,24]).

Theorem A.5 (First Kronecker limit formula). *Let γ be the Euler constant. Then, in a neighborhood of $s = 1$, we have*

$$\zeta_\tau(s; 0) = \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log(\sqrt{\tau_2}|\eta(\tau)|^2)) + \mathcal{O}(s-1),$$

where $\eta(\tau)$ is the Dedekind eta function.

Theorem A.6 (Second Kronecker limit formula). *For $v \in \mathbb{R}\backslash\mathbb{Z}$, we have*

$$\tilde{\zeta}_\tau(1, v) = -\pi \log\left|q^{1/12}\left(e\left(\frac{1}{2}v\right) - e\left(-\tfrac{1}{2}v\right)\right)\prod_{n=1}^{\infty}(1-q^n e(v))(1-q^n e(-v))\right|^2$$

with $e(v) = \mathrm{e}^{2\pi \mathrm{i} v}$ and $q = \mathrm{e}^{2\pi \mathrm{i} r}$.

From Theorems A.4–A.6, one can directly deduce the following formulas.

Lemma A.7. *Let τ and v be as above. Then*

$$\frac{\partial}{\partial s}\zeta_\tau(s; 0)|_{s=0} = -\log(4\pi^2\tau_2|\eta(\tau)|^4)$$

and

$$\frac{\partial}{\partial s}\zeta_\tau(s; v)|_{s=0} = -\log\left|q^{1/12}\left(e\left(\frac{1}{2}v\right) - e\left(-\frac{1}{2}v\right)\right)\prod_{n=1}^{\infty}(1-q^n e(v))(1-q^n e(-v))\right|^2$$

for $v \in \mathbb{R}\backslash\mathbb{Z}$. Furthermore, we have $\zeta_\tau(0; 0) = -1$ and $\zeta_\tau(0; v) = 0$.

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